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COMPOSITE MATERIALS WORKBOOK

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AIR FORCE MATERIALS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

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Stephen W. Tsai

STEPHEN W. TSAI
Project Scientist

FOR THE COMMANDER

Jerome M. Kelble

JEROME M. KELBLE, Chief
Nonmetallic Materials Division

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This workbook is intended to present to the users of composite materials a set of tools to solve most commonly encountered problems in design and testing. Attempts are made to simplify both the operational and conceptual aspects. Subjects are selected from the standpoint of practical application rather than elegance.

This workbook is unique in that it requires the student to work through many numerical problems immediately after the presentation of formulas. Program-

20. ABSTRACT (continued)

mable pocket calculators, preferably with magnetic card capability, are most suitable to perform the necessary calculations. In a separate volume, a number of the formulas in this workbook have been preprogrammed. The description, operating instruction, and program listing for these formulas have been compiled for Texas Instruments SR-52.

It is envisioned that this workbook is suitable for both a supervised educational program for the novice, and a refresher course for the experienced. It is believed that composite materials can be made conceptually simple. Operational aspects can also be made simple by the use of programmable calculators. The performance characteristics of composite materials can now be fully appreciated and utilized.

The combined workbook and mc^2 (magnetic card calculator) approach, can make speed teaching and speed learning possible. This approach can be extended into subjects beyond composite materials.

9. PERFORMING ORGANIZATION NAME AND ADDRESS (continued)

University of Dayton Research Institute
300 College Park Avenue
Dayton, Ohio 45469

11. CONTROLLING OFFICE NAME AND ADDRESS (continued)

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FOREWORD

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SECTION I

MATERIALS PROPERTIES

1. DENSITY OF COMPOSITES

a. Mass Fractions

Based on conservation of mass, the rule-of-mixtures mass-fraction relation always hold:

$$M_1 + M_2 + \cdots M_n \equiv \sum M_i = M \qquad (1)$$

$$\frac{\sum M_i}{M} = \sum m_i \equiv 1 \qquad (2)$$

$$\boxed{M_1}\boxed{M_2}\boxed{\cdots}\boxed{M_n} = M$$

$$\boxed{m_1}\boxed{m_2}\boxed{\cdots}\boxed{m_n} = 1$$

Figure 1 Mass fractions must always add up in a composite with n phases.

b. Volume Fractions

Based on existence of definable constituent volumes, the rule-of-mixtures volume-fraction relation is assumed to be valid:

$$V_1 + V_2 + \cdots V_n \doteq \sum V_i = V \qquad (3)$$

$$\frac{\sum V_i}{V} = \sum v_i = 1 \qquad (4)$$

$$\boxed{V_1}\boxed{V_2}\boxed{\cdots}\boxed{V_n} = V$$

$$\boxed{v_1}\boxed{v_2}\boxed{\cdots}\boxed{v_n} = 1$$

Figure 2 Volume fractions need not always add up for many reasons; phase boundary not defined, phase interactions due to mixing and binding, voids, etc.

c. Mass and Volume Fractions for Density

$$\rho = \frac{M}{V} \left\{ \begin{aligned} &= \frac{\sum M_i}{V} = \frac{\sum \rho_i V_i}{V} = \sum \rho_i v_i = \rho_1 v_1 + \rho_2 v_2 + \cdots \rho_n v_n && (5) \\ &\doteq \frac{\sum M_i}{\sum V_i} = \frac{\sum \rho_i V_i}{\sum \frac{M_i}{\rho_i}} = \frac{V}{M}\,\frac{\sum \rho_i v_i}{\sum \frac{m_i}{\rho_i}} = \left[\frac{\sum \rho_i v_i}{\sum \frac{m_i}{\rho_i}}\right]^{1/2} = \left[\frac{\rho_1 v_1 + \rho_2 v_2 + \cdots \rho_n v_n}{\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \cdots \frac{m_n}{\rho_n}}\right]^{1/2} && (6) \end{aligned} \right.$$

$$\doteq \frac{M}{\sum V_i} = \frac{M}{\sum \frac{M_i}{\rho_i}} = \frac{1}{\sum \frac{m_i}{\rho_i}} = \frac{1}{\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \cdots \frac{m_n}{\rho_n}} \qquad (7)$$

All equations are equally applicable and are limited by the basic assumption that

$$M_i = \rho_i V_i \tag{8}$$

holds within each constituent phase.

Equation 7 can be rewritten as follows:

$$\frac{1}{\rho} = \sum \frac{m_i}{\rho_i} \tag{9}$$

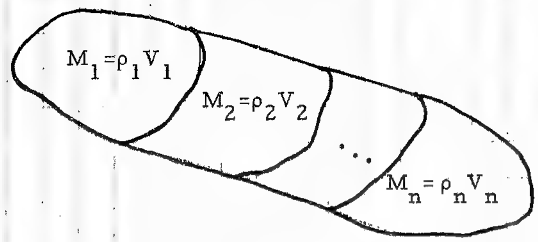


Figure 3 Rule-of-mixtures relations imply non-interacting phases, within each of which Equation 8 holds.

d. Density of Two-phase Composites

From Equation 5

$$\rho = \rho_1 v_1 + \rho_2 v_2$$

From Equation 7

$$\frac{1}{\rho} = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}$$

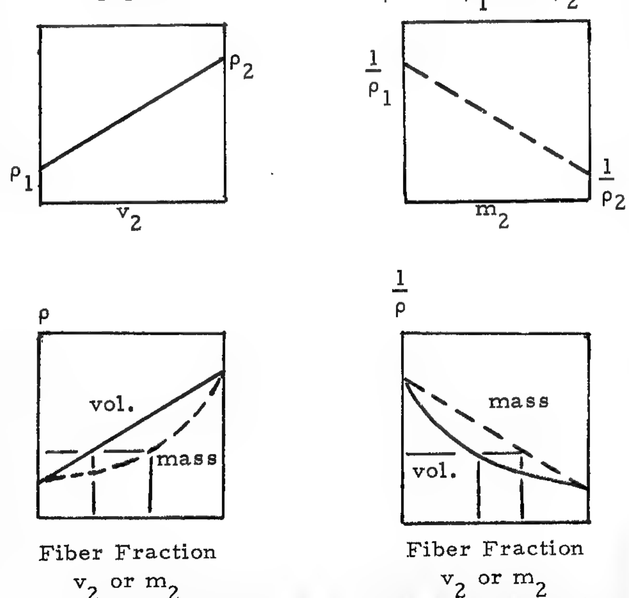


Figure 4 Linear plots are the easiest provided proper variables, ρ or $1/\rho$ is taken. Relationships between mass and volume contents are readily seen.

TABLE 1 TYPICAL CONSTITUENT DENSITIES

MATERIAL	SPECIFIC GRAVITY	DENSITY kg m^{-3}
Kevlar	1.45	1450
Graphite	1.7	1700
Glass	2.6	2600
Boron	2.6	2600
Steel	7.8	7800
W	19.3	19300
Nylon	1.1	1100
Epoxy	1.2	1200
Polyester	1.4	1400
Be	1.8	1800
Al	2.8	2800
Ti	4.5	4500

e. Void Content

Void content in a two-phase composite can be defined by

$$v_{void} = 1 - (v_1 + v_2)$$

$$= 1 - \frac{\rho_{measured}}{\rho_{calculated}}$$

$$= 1 - \frac{M}{V}\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) \qquad (10)$$

This relationship is not accurate because:

(1) The assumed non-interacting phases ignore curing stress which can induce up to 1 percent strain.

(2) Absorbed moisture which can induce swelling of several percent.

(3) Actual voids and cracks may be closed, thus, cannot influence the gross density beyond the detectable level.

Alternative methods for determination of voids will be covered later.

f. Relations between mass and volume fractions

$$v_1 = \frac{V_1}{V} = \frac{M_1}{\rho_1} \frac{\rho}{M} = \frac{m_1}{\rho_1} \rho = \frac{m_1}{\rho_1} \frac{1}{\dfrac{m_1}{\rho_1} + \dfrac{m_2}{\rho_2}} = \frac{1}{1 + \left(\dfrac{1}{m_1} - 1\right) \dfrac{\rho_1}{\rho_2}} \tag{11}$$

Composites	ρ_m / ρ_f
Gr / Ep	.71
Gl / Ep	.46
B / Ep	.46
B / Al	1.08

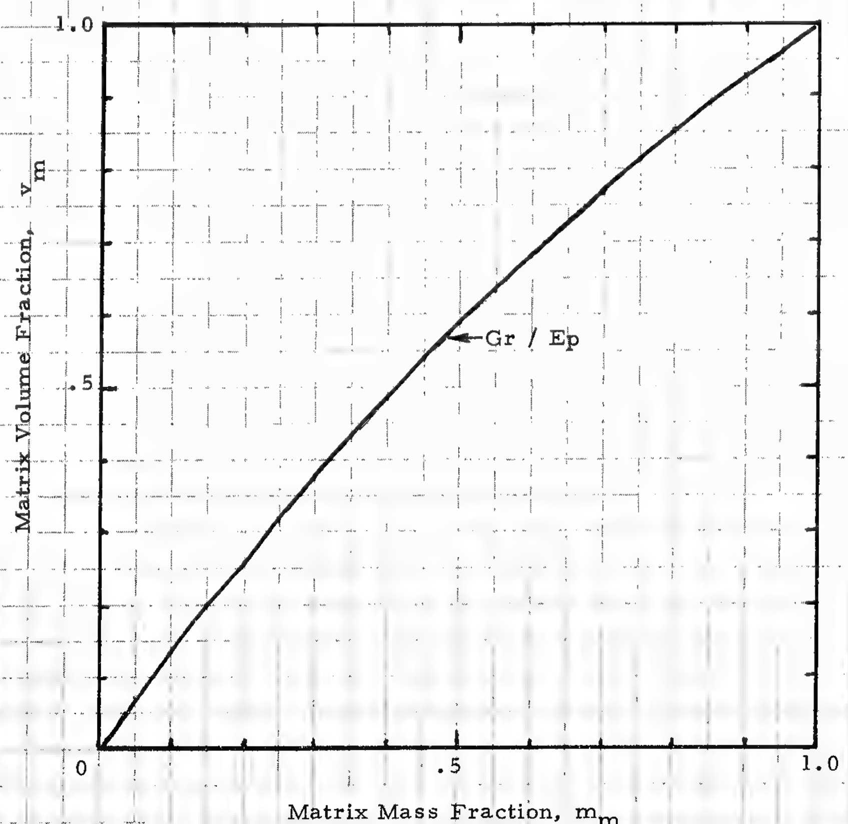


Figure 5 Mass and Volume Fraction Curve

2. ONE-DIMENSIONAL ELASTICITY PROPERTY

a. Modulus vs Compliance

$$E = \text{Modulus} = \frac{\sigma}{\epsilon} = \frac{\text{Stress}}{\text{Strain}} \qquad (12)$$

$$S = \text{Compliance} = \frac{\epsilon}{\sigma} = \frac{\text{Strain}}{\text{Stress}} \qquad (13)$$

$$E \cdot S = 1 \qquad (14)$$

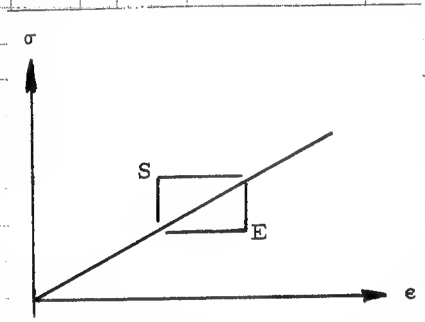
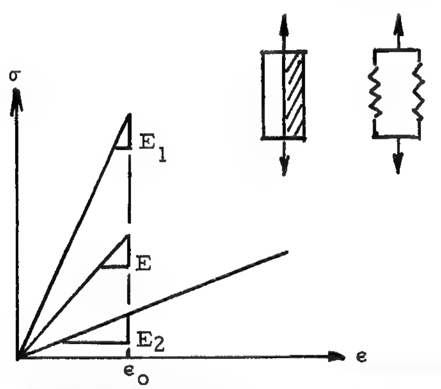


Figure 6 Modulus and compliance are reciprocal of each other. Either one can be used but one is usually preferred for a given situation. This will be illustrated on several occasions later.

b. Elasticity Property of 2 - Phase Composites



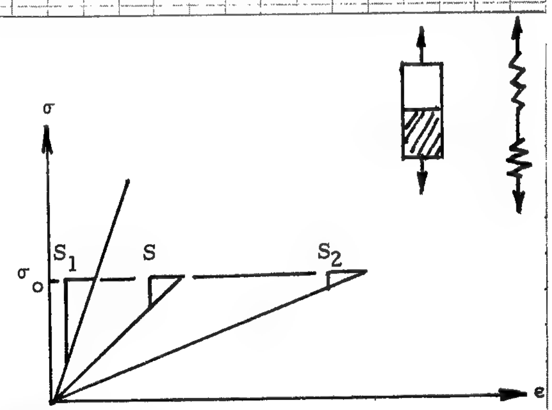


Figure 7 For a constant strain model with parallel phases, modulus E is the preferred property because simple rule of mixture equation applies:

$$E = \Sigma E_i v_i \qquad (15)$$

Figure 8 For a constant stress model with in-series phases, compliance S is the preferred property because:

$$S = \Sigma S_i v_i \qquad (16)$$

c. Graphic Illustrations of Composite Modulus and Compliance

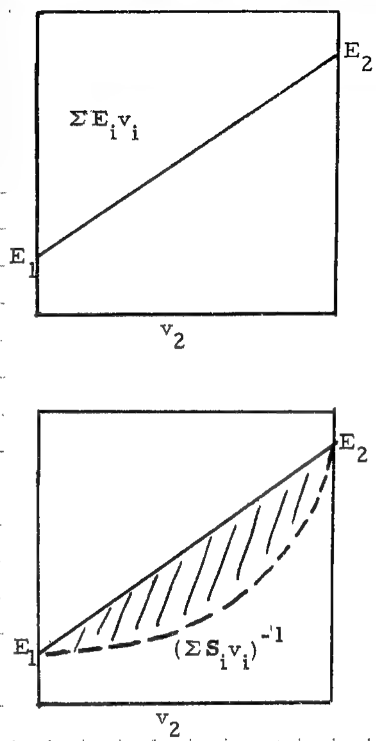


Figure 9 Bounds for composite modulus. Note lower bound is the reciprocal of composite compliance.

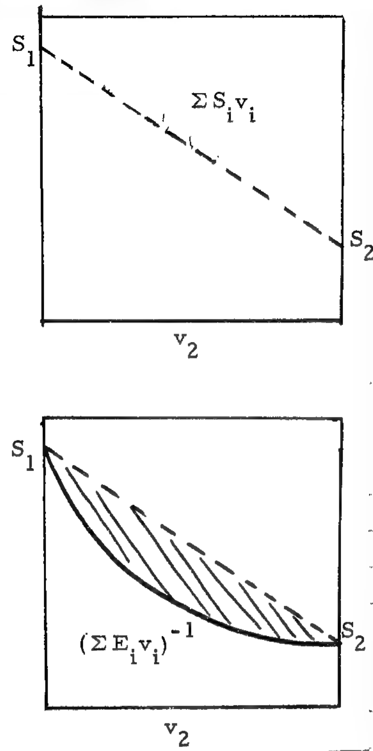


Figure 10 Bounds for composite compliance. Note lower bound is the reciprocal of composite modulus.

d. Parallel and Series Models

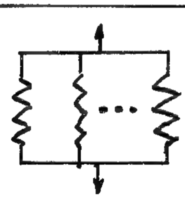
	Elasticity	Conductivity	Resistivity	Model
	$E = \Sigma E_i v_i$	$k = \Sigma k_i v_i$		"Modulus"
	$\frac{1}{S} = \sum \frac{v_i}{S_i}$		$\frac{1}{R} = \sum \frac{v_i}{R_i}$	"Compliance"
	$S = \Sigma S_i v_i$		$R = \Sigma R_i v_i$	"Modulus"
	$\frac{1}{E} = \sum \frac{v_i}{E_i}$	$\frac{1}{k} = \sum \frac{v_i}{k_i}$		"Compliance"

TABLE 2 TYPICAL CONSTITUENT STIFFNESSES

MATERIAL	YOUNG'S MODULUS E(GPa)	ACOUSTIC VELOCITY C(km sec^{-1})	
Kevlar	131	9.50	
Graphite	207	11.0	
Glass	87	5.78	
Boron	414	12.6	
Steel	207	5.15	
W	407	4.59	
Nylon	4.8	2.1	
Epoxy	3.4	1.7	
Polyester	3.4	1.6	
Be	241	11.6	
Al	69	5.0	
Ti	103	4.8	

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3. TWO-DIMENSIONAL ELASTICITY PROPERTIES

a. Strains in Terms of Stresses for Isotropic Bodies Under Plane Stress

(1) Compliance matrix in conventional form

$$e_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y$$

$$e_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y \qquad (17)$$

$$e_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

(2) Compliance matrix in tabular form

	σ_x	σ_y	σ_{xy}
e_x	$\frac{1}{E}$	$-\frac{\nu}{E}$	0
e_y	$-\frac{\nu}{E}$	$\frac{1}{E}$	0
e_{xy}	0	0	$\frac{1}{G}=\frac{2(1+\nu)}{E}$

(18)

(3) Compliance matrix in matrix form

$$\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \qquad (19)$$

(4) Compliance matrix in index form

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \qquad (20)$$

(5) Contracted Notation:

$$\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 5 \\ & 2 & 4 \\ & & 3 \end{bmatrix} \qquad (21)$$

b. Stresses in Terms of Strains for Isptropic Bodies Under Plane Stresses

(1) Modulus matrix in conventional form

$$\sigma_x = \frac{E}{1-\nu^2} e_x + \frac{\nu E}{1-\nu^2} e_y$$

$$\sigma_y = \frac{\nu E}{1-\nu^2} e_x + \frac{E}{1-\nu^2} e_y \qquad (22)$$

$$\sigma_{xy} = G\, e_{xy} = \frac{E}{2(1+\nu)} e_{xy}$$

(2) Modulus matrix in tabular form

	e_x	e_y	e_{xy}
σ_x	$\frac{E}{1-\nu^2}$	$\frac{\nu E}{1-\nu^2}$	0
σ_y	$\frac{\nu E}{1-\nu^2}$	$\frac{E}{1-\nu^2}$	0
σ_{xy}	0	0	G

(23)

(3) Modulus matrix in matrix form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} \qquad (24)$$

(4) Modulus matrix in index form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} \qquad (25)$$

Summation convention

$$\sigma_i = \Sigma Q_{ij}\, e_j = Q_{ij}\, e_j \qquad (26)$$

$$i, j = 1, 2, 6$$

c. Components of S_{ij} and Q_{ij}.

d. 3 - Dimensional Elasticity Properties for Isotropic Bodies

2D / Plane Stress

$$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} = S_{ij} \qquad \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} = Q_{ij}$$

2D / Plane Strain

$$\begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} = R_{ij} \qquad \begin{bmatrix} (1-\nu)mE & \nu mE & 0 \\ \nu mE & (1-\nu)mE & 0 \\ 0 & 0 & G \end{bmatrix} = C_{ij}$$

3D

$$\begin{bmatrix} \frac{1}{E} & \frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} = S_{ij}$$

$$\begin{bmatrix} (1-\nu)mE & \nu mE & \nu mE & 0 & 0 & 0 \\ +\nu mE & (1-\nu)mE & \nu mE & 0 & 0 & 0 \\ \nu mE & \nu mE & (1-\nu)mE & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} = C_{ij}$$

$$m = (1-\nu) / (1+\nu)(1-2\nu)$$

e. Kronecker Delta

Analogous to Equation 14, $E \cdot S = 1$

We can show that

$$Q_{ij} S_{jk} = \delta_{ik} \qquad (27)$$

where δ_{ik} = Kronecker delta $= 1$ when $i = k$
$= 0$ when $k \neq k$

SECTION II

STRESS AND STRAIN

1. STRESS AS COORDINATES ROTATE

a. Introduction

Stress is <u>NOT</u> defined as $\frac{P}{A}$ (You've flunked).

It is defined by how it changes as its reference coordinates change. That's why we need to know the ground rules. Rigid rules are particularly important for composites because composite properties also change with coordinates, by a different set of rules. All of these rules are called the transformation relations. In this section, we will discuss only those related to stress and strain.

When we apply a stress to an isotropic material, we can analyze the response of the material in the same coordinate system. There is no reason to look into any other coordinates with the possible exception of the plane where shear stress is maximum.

For composites, the response is highly dependent on the orientation of the material. It is therefore important to know how an applied stress can be transformed to the material axes. (See Figure 11) We use stress transformation relations to do this job.

Conversely, if we know the stresses in the material system x-y, we want to know what apply stresses must be for any coordinate system, we simply rotate the θ backward, or apply inverse transformation. The same rule applies except now we use negative θ in the formula.

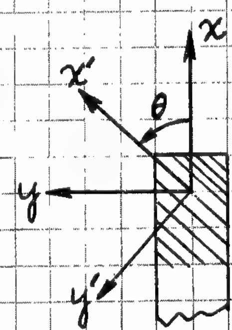

Figure 11 Transformation of applied stress in coordinate x-y to material coordinate x'-y' by a positive rotation of θ.

It is therefore an important rule of transformation to know the positive from the negative. We are all aware of the difference between tension and compression, but we rarely pay attention to positive from negative shears because they are not important for isotropic materials. For composites, we must make sure the proper signs for all angles of rotation, stresses, etc., are used. For rotation, we will use the right handed system.

b. Formula for Stress Transformation

TABLE 3 STRESS TRANSFORMATION RELATIONS

	I	R
σ'_x	1	$\cos 2(\theta - \delta)$
σ'_y	1	$-\cos 2(\theta - \delta)$
σ'_{xy}	0	$-\sin 2(\theta - \delta)$

$$I = \frac{\sigma_x + \sigma_y}{2} \tag{28}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} \tag{29}$$

$$\tan 2\delta_1 = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y} \tag{30}$$

$\delta = \delta_1 + 90$ if $\sigma_x < \sigma_y$; $\delta = \delta_1$ if $\sigma_x > \sigma_y$

Both I and R are invariants, as shown in the Mohr's Circle.

Special orientations:

(1) When $\theta = \delta$, i.e., Principal Directions

σ'_x, σ'_y = maximum or minimum

$\sigma'_{xy} = 0$

(2) When $\theta - \delta = \pm \pi/4$, i.e., max. shear orientation

$\sigma'_x = \sigma'_y = I$

σ'_{xy} = max. = R

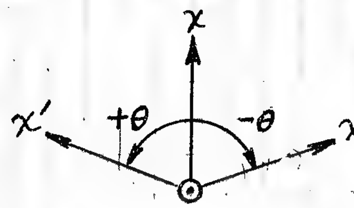

Figure 12 New coordinates x'- y' in terms of old coordinates x-y, or given x-y and θ, find x'-y'. Arrow of rotation is pointing up. If θ is negative, new coordinates are x''-y''.

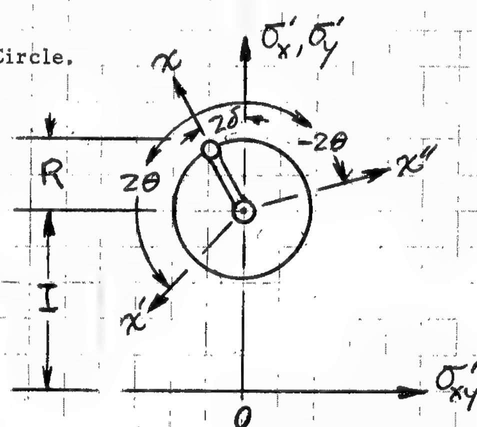

Figure 13 Mohr's Circle is defined by invariants I and R. Phase angle δ is defined by a specific combination of stress components σ_x, σ_y and σ_{xy}. As reference coordinates change by +θ, the rotation in Mohr's Circle is 2θ. Special attention should be given to value of δ:

$$\delta = \begin{cases} \delta_1 & \text{if } \sigma_x > \sigma_y \\ \delta_1 + 90 & \text{if } \sigma_x < \sigma_y \end{cases}$$

TABLE 4 ALTERNATIVE FORMULA FOR STRESS TRANSFORMATION

	σ_x	σ_y	σ_{xy}
σ'_x	m^2	n^2	$2mn$
σ'_y	n^2	m^2	$-2mn$
σ'_{xy}	$-mn$	mn	$m^2 - n^2$

$m = \cos\theta$, $n = \sin\theta$

c. Graphical Illustrations of Trigonometric Relations

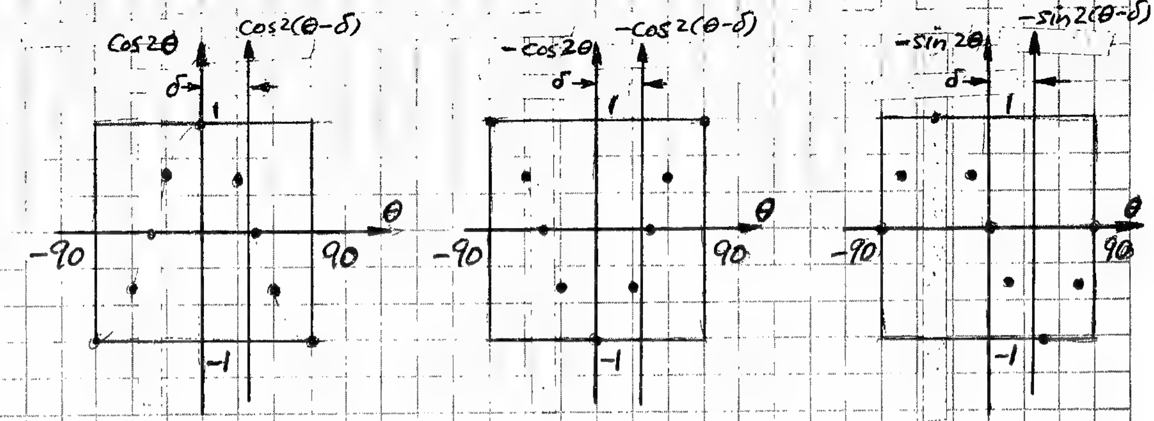

Figure 14 Relevant trigonometric functions for stress transformations. Phase angle δ displaces the function to the right (vertical axis to the left).

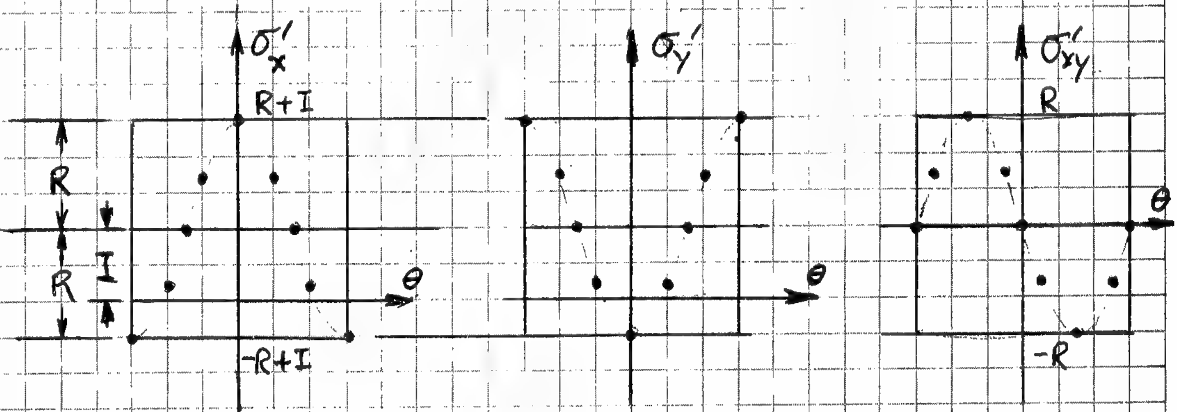

Figure 15 Typical stress transformation. Note principal stresses and maximum shear.

d. Numerical Examples

(1) Given $\sigma_i = \left\{ \quad \right\}$ (MPa)

I R δ

σ'_x σ'_y σ'_{xy}

TABLE 5 TRANSFORMED AND PRINCIPAL STRESSES

σ'_x σ'_y σ'_{xy}							

(2) Given $\sigma_i = \left\{ \quad \right\}$ (MPa)

$I =$ $\qquad$ $R =$ $\qquad$ $-\delta =$

σ'_x $\qquad$ σ'_y $\qquad$ σ'_{xy}

TABLE 6 TRANSFORMED AND PRINCIPAL STRESSES

σ'_x σ'_y σ'_{xy}						

2. MAXIMUM STRESS FAILURE CRITERION FOR UNIDIRECTIONAL COMPOSITES

a. Off-Axis Tensile Failure Stresses Under Uniaxial Stress $\sigma_x (\sigma_y = \sigma_{xy} = 0)$. The stresses in the material symmetry axis x' (along the fibers) can be obtained directly from Table 4.

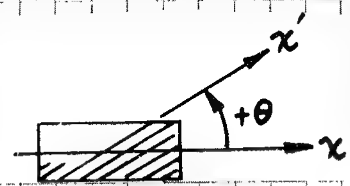

Figure 16 Coordinates for off-axis uniaxial tests. Shear stress will be positive if fibers oriented along $-\theta$.

$$\begin{aligned} \sigma'_x &= \sigma_L = \text{Longitudinal stress} = m^2 \sigma_x \\ \sigma'_y &= \sigma_T = \text{Transverse stress} = n^2 \sigma_x \\ \sigma'_{xy} &= \sigma_{LT} = \text{Shear stress LT} = -mn\sigma_x \end{aligned} \tag{31}$$

Maximum stress criterion assumes that failure will occur when lowest of the following 3 σ_x's is reached.

$$\sigma_x = \begin{cases} \dfrac{\sigma_L|\max}{m^2} = \dfrac{X}{m^2} \\[2ex] \dfrac{\sigma_T|\max}{n^2} = \dfrac{Y}{n^2} \\[2ex] \dfrac{\sigma_s|\max}{-mn} = \dfrac{S'}{mn} \end{cases} \tag{32}$$

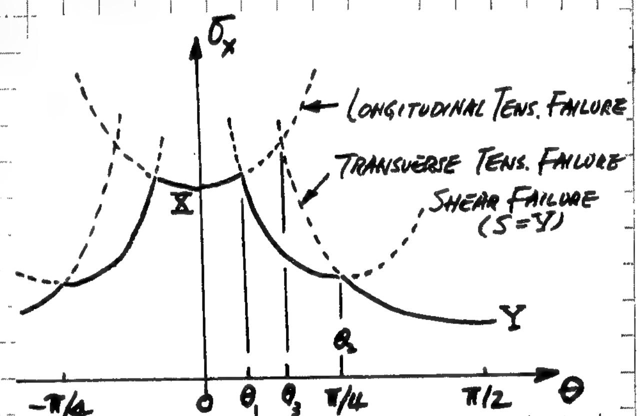

Figure 17 Maximum stress criterion is typified by separate branches which may be related to fiber and matrix (transverse or shear) failures. There is no interaction.

b. Off-Axis Compressive Uniaxial Strength

$$\sigma'_x = \begin{cases} \dfrac{X'}{m^2} \\[2ex] \dfrac{Y'}{n^2} \\[2ex] \dfrac{S}{-mn} \end{cases} \tag{33}$$

Shear strength S is assumed to be equal in positive and negative shear stresses.

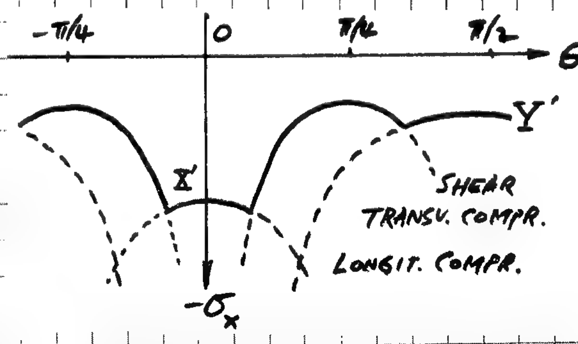

Figure 18 Maximum stress criterion as applied to off-axis compressive tests. There is no interaction between tensile and compressive strength.

c. Off-Axis Shear Strength

Substituting $\sigma_x = \sigma_y = 0$ into Table 4 (for $\pm\theta$),

$$\left.\begin{aligned} \sigma'_x &= 2mn\,\sigma_{xy} = (\sin 2\theta)\sigma_{xy} \\ \sigma'_y &= -2mn\,\sigma_{xy} = -(\sin 2\theta)\sigma_{xy} \\ \sigma'_{xy} &= (m^2-n^2)\sigma_{xy} = (\cos 2\theta)\sigma_{xy} \end{aligned}\right\} \quad (34)$$

Depending on the signs of the stress components σ'_x, σ'_y, σ'_{xy}, which are functions of the signs of θ and σ_{xy}, the following strength criteria shall be used. Note: Each quadrant in the table has distinct combinations.

	$-\theta$	$+\theta$
$+\sigma_{xy}$	$X'/\sin 2\theta$ $Y/\sin 2\theta$	$X/\sin 2\theta$ $Y'/\sin 2\theta$
$-\sigma_{xy}$	$X/\sin 2\theta$ $Y'/\sin 2\theta$	$X'/\sin 2\theta$ $Y/\sin 2\theta$

	$\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$	$-\frac{3}{4}\pi \le \theta \le -\frac{1}{4}\pi$ $\frac{1}{4}\pi \le \theta \le \frac{3}{4}\pi$
$+\sigma_{xy}$	$S/\cos 2\theta$	$S'/\cos 2\theta$
$-\sigma_{xy}$	$S'/\cos 2\theta$	$S/\cos 2\theta$

Numerous failure criteria other than the maximum stress will be discussed later.

d. The intersection of any 2-failure curves in Figure 17 are as follows:

$$\theta_1 = \cot^{-1} \frac{X}{S'} \quad (35)$$

$$\theta_2 = \tan^{-1} \frac{Y}{S'} \quad (36)$$

$$\theta_3 = \cot^{-1} \sqrt{\frac{X}{Y}} \quad (37)$$

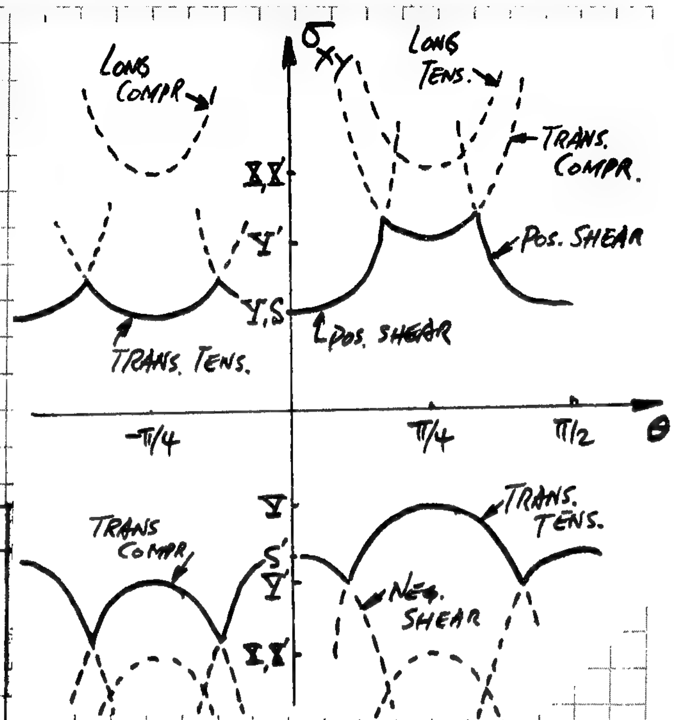

Figure 19. Off-axis shear strength based on maximum stress criterion. Six possible failure modes, including positive and negative shears are labeled. In this case, longitudinal tensile and compressive are never limiting cases. Matrix tensile, compressive and shear are the controlling modes.

3. STRENGTH ESTIMATE OF RANDOM FIBER COMPOSITES

a. Tensile Strength

It may be assumed that the strength of a random fiber composite is equal to the average strength of the area under the off-axis tensile strength of the area under the off-axis tensile strength of a unidirectional composite with the same constituents, (Figure 14).

$$\bar{X} = \frac{2}{\pi}\int_0^{\pi/2} X_\theta \, d\theta \tag{38}$$

$$= \frac{2}{\pi}\left[X\int_0^{\theta_1}\frac{d\theta}{\sin^2\theta} + S\int_{\theta_1}^{\theta_2}\frac{d\theta}{\sin\theta\cos\theta} + Y\int_{\theta_2}^{\pi/2}\frac{d\theta}{\cos^2\theta}\right] \tag{39}$$

After integration -

$$\frac{\bar{X}}{Y} = \begin{cases} \dfrac{4}{\pi}\,a\left[1+\dfrac{1}{2}\sqrt{\dfrac{X}{a^2 Y}}\right], & a \le \sqrt{\dfrac{X}{Y}} \quad (40) \\[2ex] \dfrac{4}{\pi}\sqrt{\dfrac{X}{Y}}, & a > \sqrt{\dfrac{X}{Y}} \quad (41) \end{cases}$$

Where $a = \dfrac{S}{Y}$

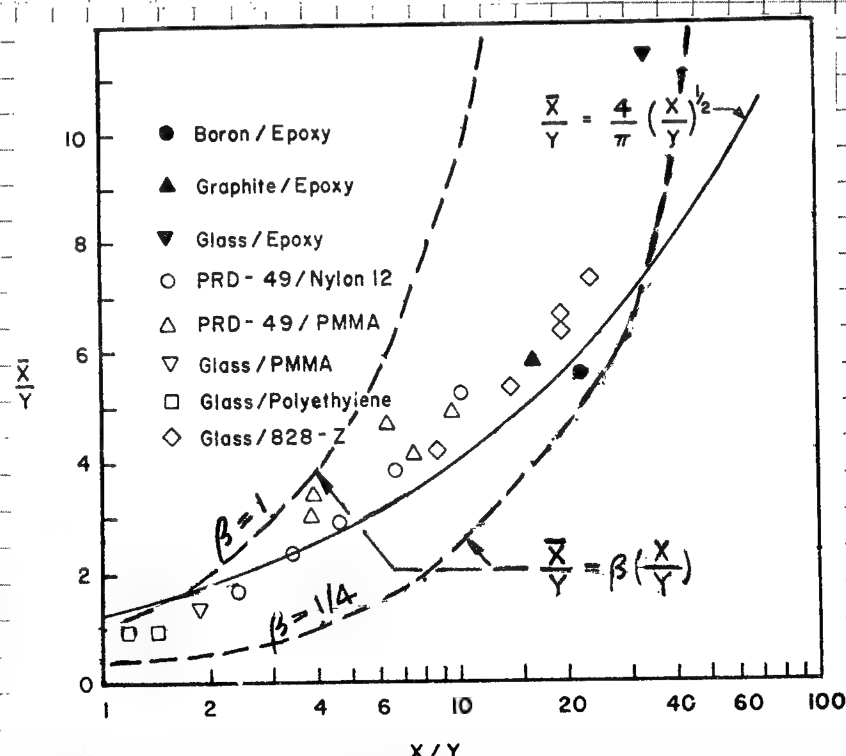

Figure 20 The analytic estimate (based on high shear strength) falls below most available data by a factor of 1.5; but the trend appears to be in general agreement. The rule-of-mixtures equation, with correction factor β, is shown in dashed lines. [1]

b. Compressive Strength

c. Shear Strength

d. Strengths of Particulate Composites

4. STRAIN AS COORDINATES ROTATE

a. Formula for Strain Transformation

Only the shear component is the difference between the strain and stress transformation relations, i.e.,

TABLE 7 STRAIN TRANSFORMATION RELATIONS

	I	R
e'_x	1	$\cos 2(\theta - \delta)$
e'_y	1	$-\cos 2(\theta - \delta)$
e'_{xy}	0	$-2\sin 2(\theta - \delta)$

$$I = \frac{e_x + e_y}{2} \tag{42}$$

$$R = \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{e_{xy}}{2}\right)^2} \tag{43}$$

$$\tan 2\delta_1 = \frac{e_{xy}}{e_x - e_y} \tag{44}$$

$\delta = \delta_1$ if $e_x > e_y$

$\delta = \delta_1 + 90$ if $e_x < e_y$

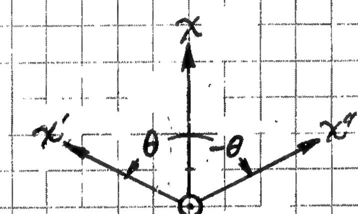

Figure 21 Positive and negative rotations for transformation.

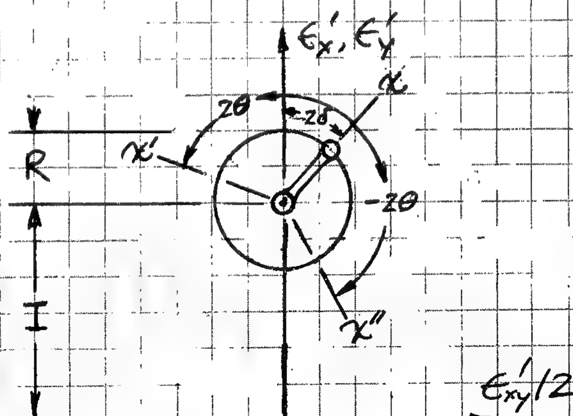

Figure 22 Mohr's Circle representation of strain transformation. Only the one-half factor for the shear strain is the difference between the strain representation and that for stress. Special attention must be paid to starting point: $\delta = \delta_1$, if $e_x > e_y$; $\delta = \delta_1 + 90$, if $\epsilon_x < \epsilon_y$.

b. Graphical Illustration of Trigonometric Rotations

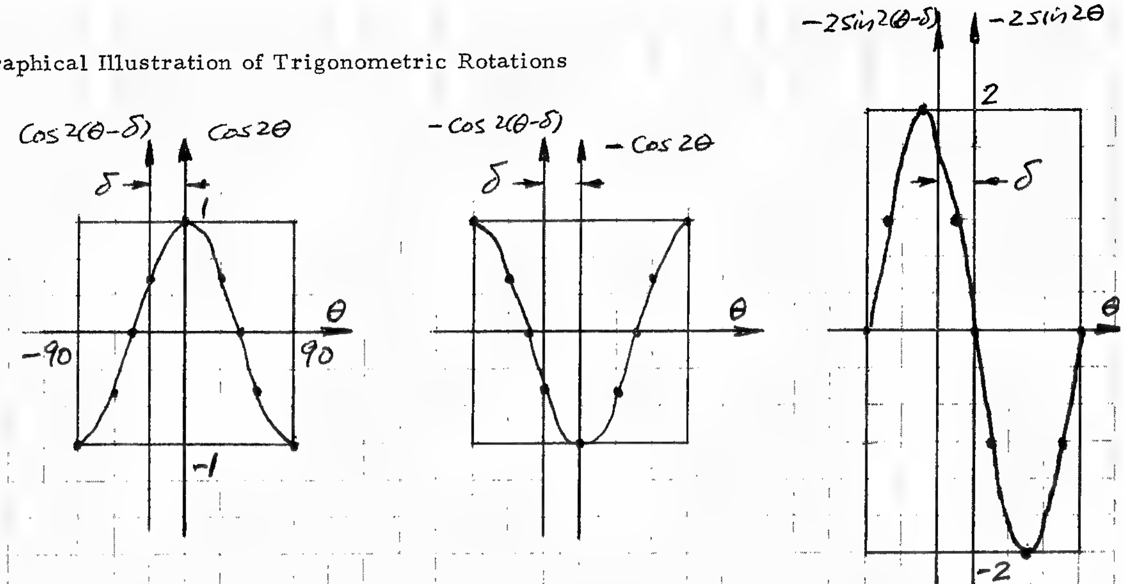

Figure 23 Relevant trigonometric functions and negative phase angle that would displace the functions to the left (or vertical axis to the right).

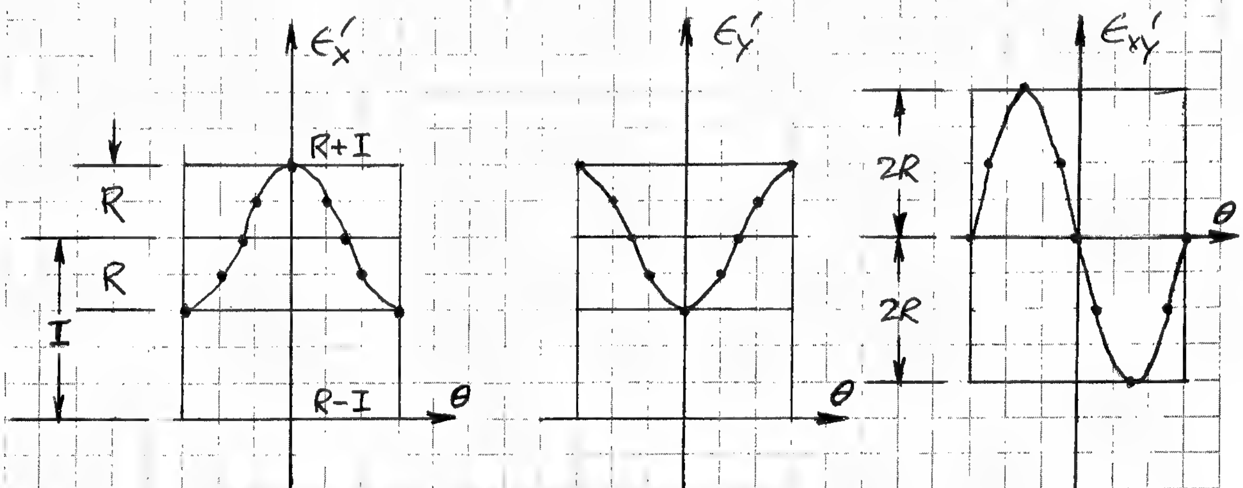

Figure 24 Typical strains transformations, note, principal strain and maximum shear strain.

c. Numerical Examples

(1) Given $e_i = \left\{ \quad \right\}$ (mm/m) or (10^3 µm/m)

$I =$ $\qquad$ $R =$ $\qquad$ $\delta =$

e'_x $\qquad$ e'_y $\qquad$ e'_{xy}

TABLE 8 TRANSFORMED AND PRINCIPAL STRAINS

e'_x						
e'_y						
e'_{xy}						

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(2) Given $e_i = \left\{ \quad \right\}$ (mm/m) or ($10^3 \mu m/m$)

$I =$ $R =$ $\delta =$

e'_x | e'_y | e'_{xy}

TABLE 9 TRANSFORMED AND PRINCIPAL STRAINS

e'_x e'_y e'_{xy}						

5. STRAIN ROSETTES

a. Three-Element Rosettes

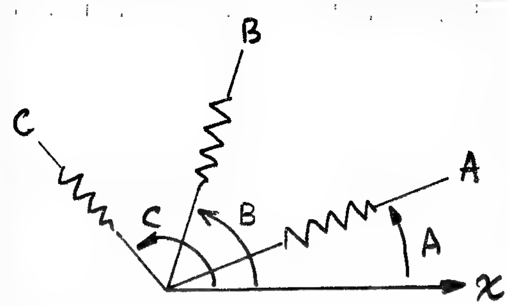


Figure 25 Strain rosette orientation with positive angles. If an angle is negative, it shall be so entered into Equations (45) and (46).

Since there are 3 strain components at each point, 3-element rosettes are in general needed to solve for 3 unknowns. Assuming 3 elements are mounted at 3 different angles from some reference coordinates, each rosette must satisfy

$$
\begin{aligned}
e_A &= (\cos^2 A)e_x + (\sin^2 A)e_y + (\sin A \cos A)e_{xy} \\
e_B &= (\cos^2 B)e_x + (\sin^2 A)e_y + (\sin B \cos B)e_{xy} \qquad (45) \\
e_C &= (\cos^2 C)e_x + (\sin^2 C)e_y + (\sin C \cos C)e_{xy}
\end{aligned}
$$

In matrix form

$$
\begin{bmatrix} \cos^2 A & \sin^2 A & \sin A \cos A \\ \cos^2 B & \sin^2 B & \sin B \cos B \\ \cos^2 C & \sin^2 C & \sin C \cos C \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = \begin{Bmatrix} e_A \\ e_B \\ e_C \end{Bmatrix} \qquad (46)
$$

b. Other Rosettes

Four-element rosette is an over-determined system when a 4th equation is added:

$$
e_D = (\cos^2 D)e_x + (\sin^2 D)e_y + (\sin D \cos D)e_{xy} \qquad (47)
$$

Methods of solution for an over-determined (4 or more equations for 3 unknowns) are available. The additional strain gage also serves as a redundant gage, in case of a defective gage.

Two-element rosette is adequate if additional information on the strain components is given. For example, if from symmetry considerations, e_{xy} is known to be zero, or $e_x = e_y$, two-element rosette can be used. The angle between the elements (i.e., A-B) can be any value, although normally it is 90°.

One-element rosette is adequate if it is known that $e_x = e_y = 0$, $e_{xy} \neq 0$; or only uniaxial strain is needed.

c. Sample Problems.

(1) From a uniaxial test,

$A = 0, B = 60^{o}, C = -60^{o}$, and $\epsilon_A, \epsilon_B, \epsilon_C = 10, -3, -3$ mm/m,

find Poisson's ratio along a-axis.

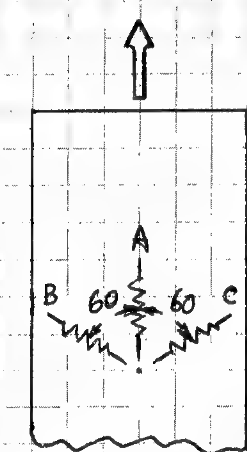
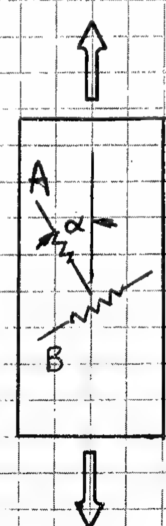


Figure 26. Rosette for uniaxial test.

(2) Find off-axis mounted 2-element orthogonal rosette in order to achieve full 15 mm/m range. Assume ultimate longitudinal strain of 3 percent and Poisson's ratio of 0.3. Is there an optimum orientation?



SECTION III

STRESS - STRAIN RELATIONS

1. COMPLIANCE AND MODULUS MATRICES

a. Stress-Strain Relations in Longhand Form (Plane Stress)

$$e_i = S_{ij}\sigma_j , \qquad i, j = 1, 2, 6 \tag{48}$$

$$\begin{aligned}
e_1 &= S_{1j}\sigma_j = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{16}\sigma_6 \\
e_2 &= S_{2j}\sigma_j = S_{21}\sigma_1 + S_{21}\sigma_2 + S_{26}\sigma_6 \\
e_6 &= S_{6j}\sigma_j = S_{61}\sigma_1 + S_{62}\sigma_2 + S_{66}\sigma_6
\end{aligned} \tag{49}$$

$$\sigma_i = Q_{ij}e_j \tag{50}$$

$$\begin{aligned}
\sigma_1 &= Q_{1i}e_i = Q_{11}e_1 + Q_{12}e_2 + Q_{16}e_6 \\
\sigma_2 &= Q_{2i}e_i = Q_{21}e_1 + Q_{22}e_2 + Q_{26}e_6 \\
\sigma_6 &= Q_{6i}e_i = Q_{61}e_1 + Q_{62}e_2 + Q_{66}e_6
\end{aligned} \tag{51}$$

b. In Matrix Form

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{21} & S_{22} & S_{26} \\ S_{61} & S_{62} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{52}$$

S_{ij} is symmetric, i.e., $S_{ij} = S_{ji}$, or $S_{12} = S_{21}$, $S_{16} = S_{61}$, $S_{26} = S_{62}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} \qquad (53)$$

Q_{ij} is also symmetric, i.e., $Q_{ij} = Q_{ji}$, or $Q_{12} = Q_{21}$, $Q_{16} = Q_{61}$, $Q_{26} = Q_{62}$

c. Elastic Symmetries

TABLE 10 COMPLIANCE AND MODULUS IN TERMS OF ELASTIC SYMMETRIES

Symmetry (No Indep. Const)	Compliance Matrix S_{ij} $(TPa)^{-1}$	Modulus Matrix Q_{ij} (MPa)
Anisotropic (6) or Generally Orthotropic (4)	$\begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & S_{16} \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & S_{26} \\ S_{61} & S_{62} & \frac{1}{G_{12}} \end{bmatrix}$	$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix}$
Specially Orthotropic (4)	$\begin{bmatrix} \frac{1}{E_L} & -\frac{\nu_{LT}}{E_L} & 0 \\ -\frac{\nu_{TL}}{E_T} & \frac{1}{E_T} & 0 \\ 0 & 0 & \frac{1}{G_{LT}} \end{bmatrix}$	$\begin{bmatrix} mE_L & \nu_{TL} mE_L & 0 \\ \nu_{LT} mE_T & mE_T & 0 \\ 0 & 0 & G_{LT} \end{bmatrix}$
Isotropic (2)	$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix}$	$\begin{bmatrix} mE & \nu mE & 0 \\ \nu mE & mE & 0 \\ 0 & 0 & G \end{bmatrix}$

For orthotropic material: $m = \frac{1}{1-\nu_{LT}\nu_{TL}} = \frac{1}{1-\nu_{LT}^2 \frac{E_T}{E_L}}$ (54)

$$\nu_{LT}E_T = \nu_{TL}E_L$$

For isotropic material: $m = \frac{1}{1-\nu^2}$ (55)

$$G = \frac{E}{2(1+\nu)}$$

Note: Q_{ij} can be expressed in terms of engineering constants only for orthotropic and isotropic materials, not for anisotropic materials. S_{ij} does not have such limitations.

TABLE 11 ENGINEERING CONSTANTS IN TERMS OF S_{ij} & Q_{ij}

	S_{ij}	Q_{ij}
E_L	$\frac{1}{S_{11}}$	$Q_{11} - \frac{Q_{12}^2}{Q_{22}}$
E_T	$\frac{1}{S_{22}}$	$Q_{22} - \frac{Q_{12}^2}{Q_{11}}$
ν_{12}	$-\frac{S_{12}}{S_{11}}$	$\frac{Q_{12}}{Q_{22}}$
ν_{21}	$-\frac{S_{12}}{S_{22}}$	$\frac{Q_{12}}{Q_{11}}$
G_{LT}	$\frac{1}{S_{66}}$	Q_{66}

d. Elastic Constants

TABLE 12 ENGINEERING CONSTANTS FOR UNIDIRECTIONAL COMPOSITES

Type	Material	Fiber Vol. v_f	Specific Gravity γ	E_L GPa	E_T GPa	ν_{LT}	G_{LT} GPa
B(4) 5505	B/Ep	0.5	2.0	204	18.5	0.23	5.79
Mod II 5206	Gr/Ep	0.55	1.5	55	8.83	0.30	5.24
HMS 3002M	Gr/Ep	0.48	1.58	185	6.76	0.20	5.86
T300 5208	Gr/Ep	0.70	1.60	181	10.3	0.28	7.17
Mod I ERLA 4289	Gr/Ep	0.51	1.56	188	4.14	0.20	4.83
Mod I ERLA 4617	Gr/Ep	0.45	1.54	190	7.10	0.10	6.2
AS 3501	Gr/Ep	0.66	1.60	138	8.96	0.30	7.1
B(4) WRD 9371	B/PI	0.49	2.0	222	14.5	0.16	7.7
Mod I WRD 9371	Gr/PI	0.45	1.54	216	4.97	0.25	4.5
S Glass 1009-26-5901	Gl/Ep	0.72	2.13	60.7	24.8	0.23	12.0
Scotchply 1002	Gl/Ep	0.45	1.8	38.6	8.27	0.26	4.14
T300 SP 313	Gr/Ep	0.65	1.55	140	9.7	0.32	
Kevlar 49 Epoxy	Kev/Ep	0.60	1.38	76	5.5	0.34	2.3

TABLE 13 COMPLIANCE S_{ij} IN TERMS OF ENGINEERING CONSTANTS $(10^{12}\text{Pa})^{-1}$

Composites	$S_{11} = \frac{1}{E_L}$ $(\text{TPa})^{-1}$	$S_{22} = \frac{1}{E_T}$ $(\text{TPa})^{-1}$	$S_{12} = -\nu_{LT}S_{11}$ $= -\nu_{TL}S_{22}$	$S_{66} = \frac{1}{G_{LT}}$ $(\text{TPa})^{-1}$
T-300/5208	5.52	97.1	-1.55	139
B/5505	4.90	54.0	-1.12	173
Scotchply 1002	25.9	121.0	-6.74	242

TABLE 14 MODULUS Q_{ij} IN TERMS OF ENGINEERING CONSTANTS (10^6Pa)

Composites	m	$Q_{11} = mE_L$ (GPa)	$Q_{22} = mE_T$ (GPa)	$Q_{12} = \nu_{LT}Q_{22}$ $= \nu_{TL}Q_{11}$	$Q_{66} = G_{LT}$ (GPa)	$S_{ij}Q_{jk} = \delta_{ik}$
T-300/5208	1.004	182	10.3	2.90	7.17	
B/5505	1.005	205	18.6	4.28	5.79	
Scotchply 1002	1.005	39.2	8.39	2.18	4.14	

$$m = \frac{1}{1-\nu_{LT}\nu_{TL}} = \frac{1}{1-\nu_{LT}^2 E_T/E_L}$$

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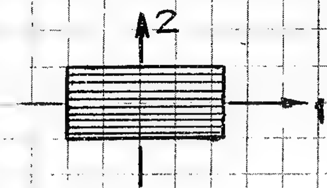
e. Determination of Strains from Stresses for a



$$\sigma_i \text{ in MPa} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} 400 \\ 60 \\ 15 \end{Bmatrix}$$

$$S_{ij} \text{ for Gr/Ep in } (\text{TPa})^{-1} = \begin{bmatrix} 5.52 & -1.55 & 0 \\ -1.55 & 97.1 & 0 \\ 0 & 0 & 139 \end{bmatrix}$$

Figure 27 Coordinates for σ_i and S_{ij} are coincident.

$$e_i \text{ in mm/m} = \begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} 2.15 \\ 5.21 \\ 2.08 \end{Bmatrix} \text{ mm/m}$$

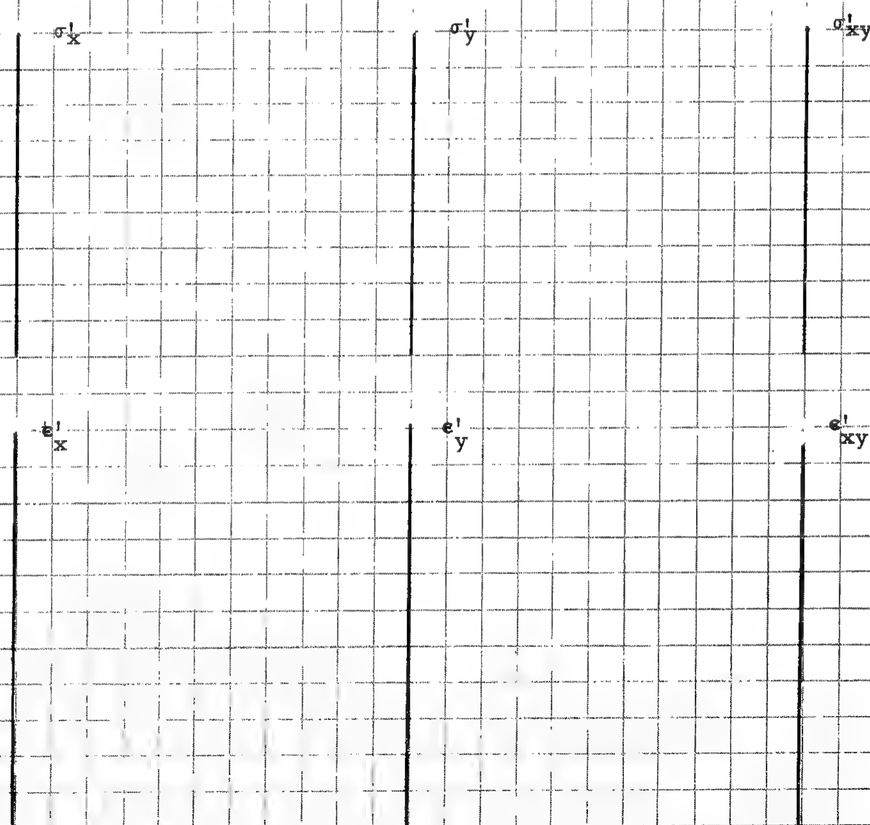


Figure 28. For a given set of Imposed Stresses, shown are the resulting strains for T-300/5208 unidirectional composites for all orientations.

f. Determination of Stresses from Strains for a Given Material

ϵ_i in mm/m
$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \end{Bmatrix}$$

Q_{ij} for Gr/Ep in GPa
$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

σ_i in MPa
$$= \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \end{Bmatrix}$$

ϵ'_x ϵ'_y ϵ'_{xy}

σ'_x σ'_y σ'_{xy}

Figure 29 For a given set of Imposed Strains, shown are the resulting stresses for T-300/5208 unidirectional composites for all orientations.

2. MAXIMUM STRAIN CRITERION

a. Determination of Ultimate Strains

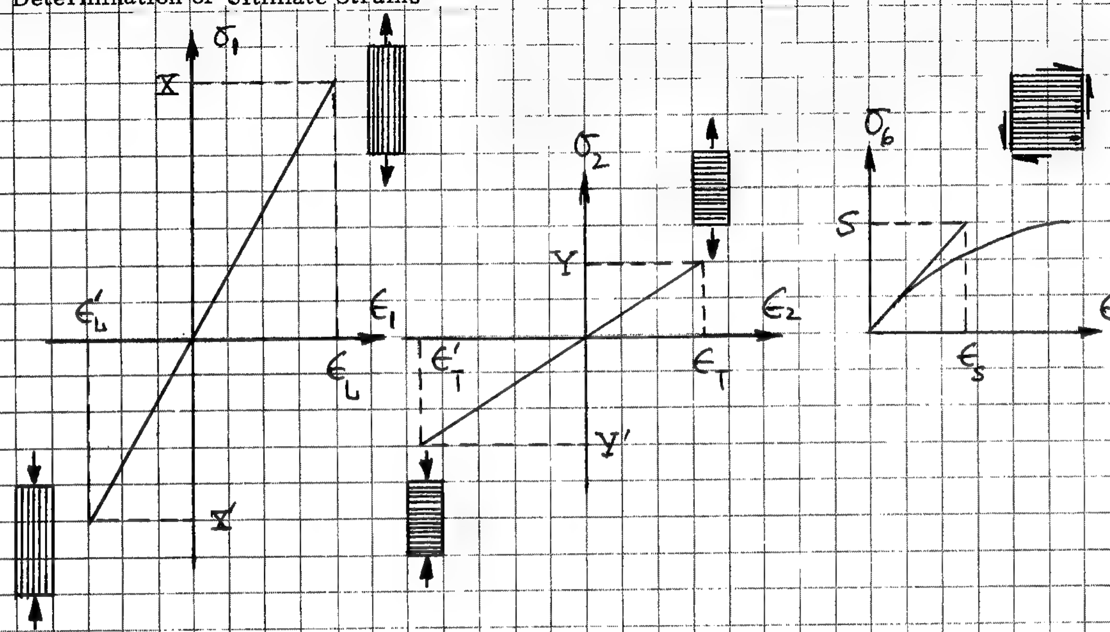


Figure 30 Ultimate stress and strain determination.

b. Typical Ultimate Strains of Unidirectional Composites

TABLE 15. ULTIMATE STRAINS OF UNIDIRECTIONAL COMPOSITES (mm/m)

Material	e_L	e'_L	e_T	e'_T	e_S
T-300/5208	8.30	8.30	3.91	23.8	9.42
B/5505	6.18	12.2	3.30	10.9	11.6
Scotchply 1002	27.5	15.8	3.80	14.2	17.4

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c. Maximum Strains Criterion in Strain Space

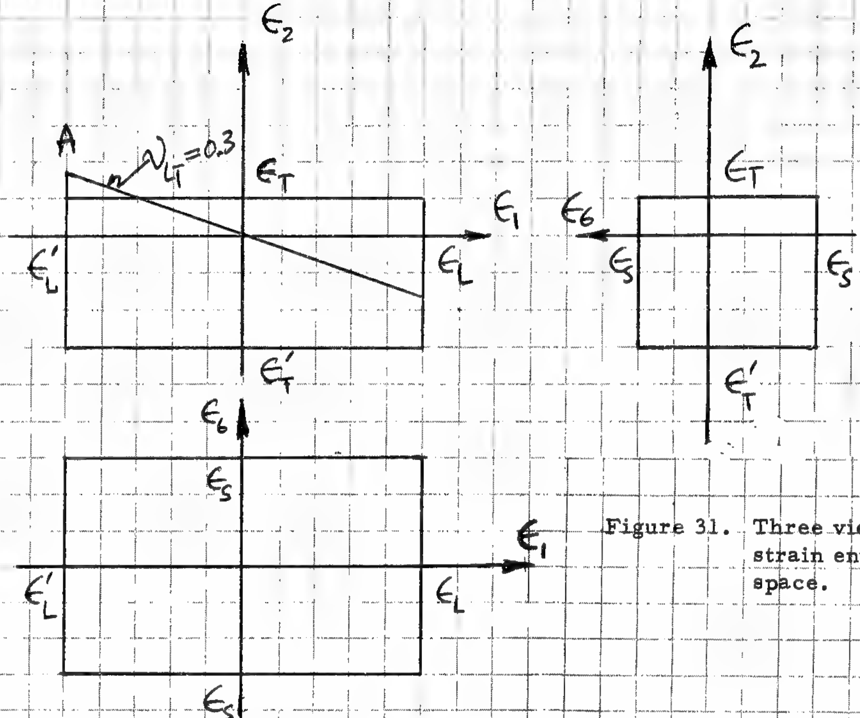


Figure 31. Three views of maximum strain envelope in strain space.

d. Limitation Imposed by Poisson Strain

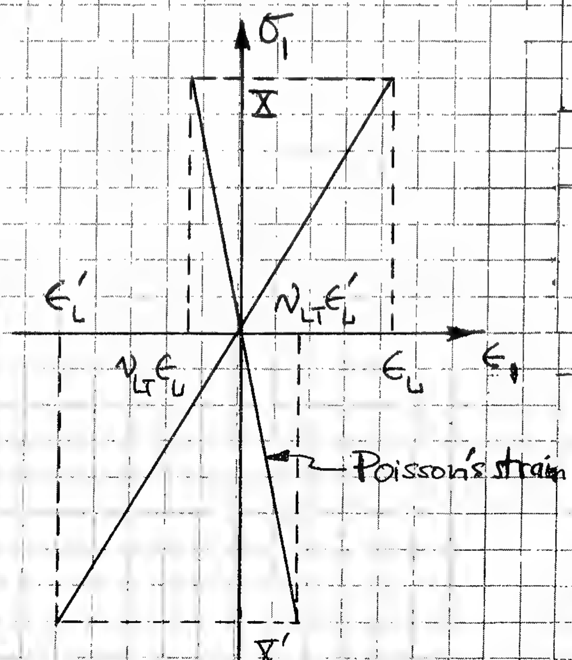


Figure 32 Longitudinal and transverse strain in uniaxial axial tension and compression tests.

If $\nu_{LT}e'_L > e_T$, strains envelope may <u>not</u> include all failure points. Point A, for example, could be outside failure surface as shown in Figure 31. This can be an inconsistency of this failure criterion.

e. Maximum Strain Criteria in Stress Space

From Stress-Strain relation $S_{ij}\sigma_j = e_i$

$$\sigma_1 - \nu_{LT}\sigma_2 \leq E_L e_L$$

$$\leq X \qquad (56)$$

$$-\sigma_1 + \nu_{LT}\sigma_2 \leq X' \qquad (57)$$

$$-\nu_{TL}\sigma_1 + \sigma_2 \leq Y \qquad (58)$$

$$\nu_{TL}\sigma_1 - \sigma_2 \leq Y' \qquad (59)$$

$$\sigma_6 \leq S \qquad (60)$$

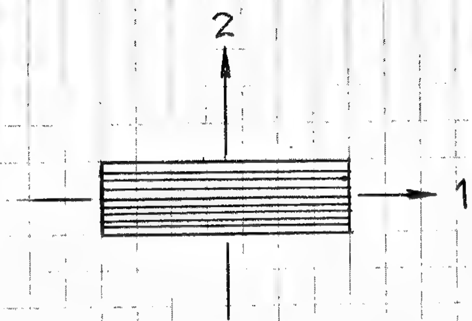
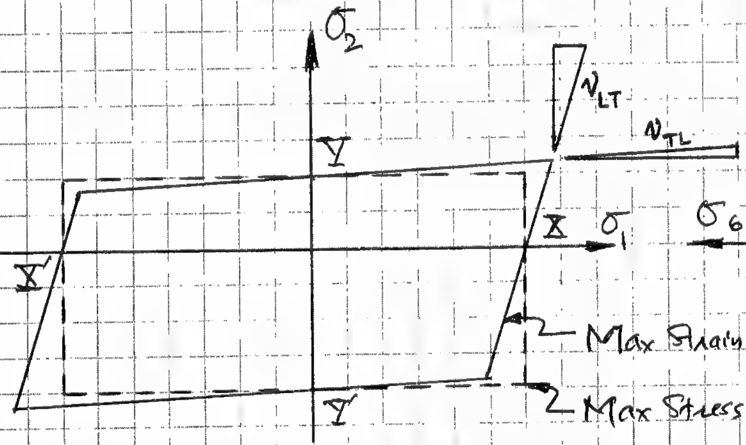
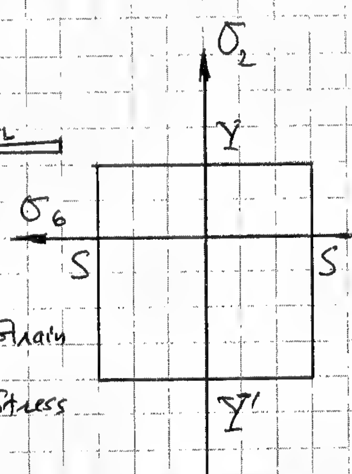


Figure 33 Strain criterion is limited to the material symmetry axes shown here.





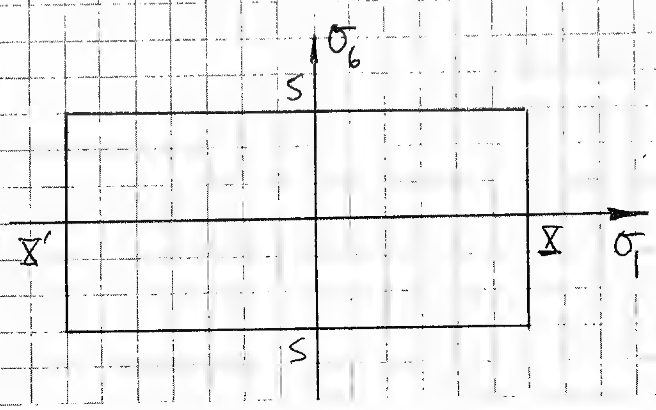


Figure 34 Three views of the maximum strain failure envelope in stress space.

f. Off-axis Uniaxial Tests

(1) Maximum Strain Criterion

Substitute

$$\sigma_1' = n^2\sigma_1 \;, \quad \sigma_2' = m^2\sigma_1 \;, \quad \sigma_6' = mn\sigma_1 \quad \text{into SSR}$$

$$\epsilon_L \geq (n^2 - \nu_{LT}m^2)\,\sigma_1/E_L \qquad (61)$$

$$\epsilon_T \geq (m^2 - \nu_{TL}n^2)\,\sigma_1/E_T \qquad (62)$$

$$\epsilon_{LT} \geq -mn\sigma_1/G_{LT} \qquad (63)$$

$$\sigma_1 \leq \begin{cases} \dfrac{X}{n^2 - \nu_{LT}m^2} \\[2ex] \dfrac{Y}{m^2 - \nu_{TL}n^2} \\[2ex] \dfrac{S}{mn} \end{cases} \qquad (64) \qquad -\sigma_1 \leq \begin{cases} \dfrac{X'}{n^2 - \nu_{LT}m^2} \\[2ex] \dfrac{Y'}{m^2 - \nu_{TL}n^2} \\[2ex] \dfrac{S'}{mn} \end{cases} \qquad (65)$$

(2) Maximum Stress Criterion

A special case of (1) when $\nu_{LT} = \nu_{TL} = 0$.

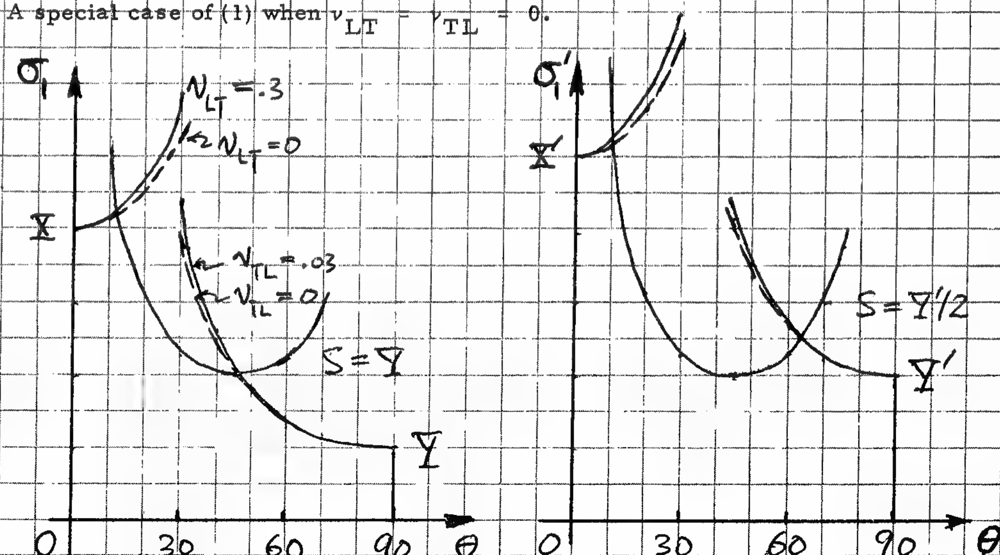


Figure 35 Difference between the predicted max stress and max strain criteria of the off-axis strength.

3. PLANE ELASTICITY SOLUTIONS

a. Complex Parameters

Stress distribution in a plane problem is a function of the characteristic equation for an orthotropic material:

$$S_{11}\mu^4 + (2S_{12} + S_{66})\mu^2 + S_{22} = 0 \tag{66}$$

In terms of engineering constants with 0° oriented along the reference coordinate:

$$\mu^4 + \left(\frac{E_L}{G_{LT}} - 2\nu_{LT}\right)\mu^2 + \frac{E_L}{E_T} = 0 \tag{67}$$

Roots of this equation are called complex parameters which constitute another measure of the elastic constants:

$$\mu = \beta i , \delta i \tag{68}$$

Numerical example for B/Ep unidirectional composites:

$$\mu^4 + \left(\frac{206.5}{6.9} - 2 \times 0.21\right) \mu^2 + \left(\frac{206.5}{18.6}\right) = 0 \tag{69}$$

$$\mu^4 + 29.1\mu^2 + 11.1 = 0 \tag{70}$$

$$\mu^2 = -14.6 \pm \sqrt{214-11.1} = -14.6 \pm 14.2 = -28.8, -.4$$

$$\therefore \beta = 5.35 , \quad \delta = .63 \tag{71}$$

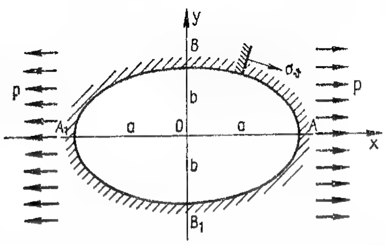
For isotropic material, $\beta = \delta = 1$

TABLE 16 COMPLEX PARAMETERS FOR VARIOUS UNIDIRECTIONAL COMPOSITES

Composites	β	δ	$n = \beta + \delta$	$k = \beta\delta$

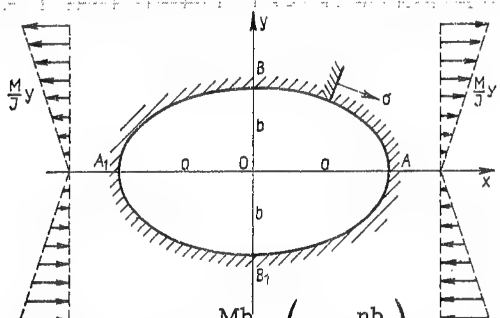
b. Stress Concentrations at Various Boundary Conditions

(1) Elliptic Hole Under Tension



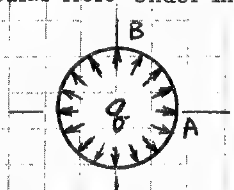
$$\sigma_\theta = p\left(1 + \frac{nb}{a}\right) \qquad (72)$$

(2) Elliptic Hole Under Bending



$$\sigma_\theta = \pm\frac{Mb}{I}\left(1 + \frac{nb}{2a}\right) \qquad (73)$$

(3) Circular Hole Under Internal Pressure

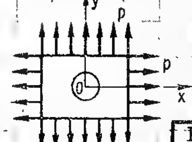


At Point A $\quad \sigma_\theta = q\,\dfrac{n-1}{k}$

(74)

At Point B $\quad \sigma_\theta = q\,(n-k)$

(4) Circular Hole Under Uniform Tension

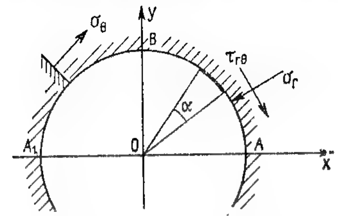


At Point A $\quad \sigma_\theta = p\sqrt{\dfrac{E_{11}}{E_{22}}}\,(n+k-1)$

(75)

At Point B $\quad \sigma_\theta = p(n-k+1)$

(5) Circular Hole Subjected to Twist

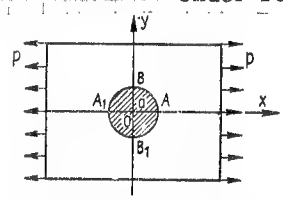


$$\sigma_r = -\frac{a}{g}\left[k-\nu_{12}+n(\cos^2\theta+\sin\theta)\right]$$

$$\tau_{r\theta} = -\frac{b}{g}\left[k-\nu_{12}+n(\cos^2\theta+n\sin^2\theta)\right]$$

$$g = \frac{1}{Q_{22}} + \frac{k}{G_{12}} \qquad (76)$$

(6) Rigid Reinforcement Under Tension



At Point A

$$\sigma_r = \frac{p}{g\sqrt{E_{11}E_{22}}}\left(k+n-\nu_{12}+\frac{E_{11}}{G_{12}}\right)$$

$$\sigma_\theta = \nu_{21}\sigma_r\,, \quad \sigma_{r\theta} = 0 \qquad (77)$$

At Point B

$$\sigma_r = \frac{p}{gE_{11}}\left[k-\nu_{12}(1+n)\right]$$

$$\sigma_\theta = \nu_{12}\sigma_r\,, \quad \sigma_{r\theta} = 0 \qquad (78)$$

General Reference:

S. G. LEKHNITSKII, Anisotropic Plates, English translation by Gordon & Breach.

c. Numerical Example of Elliptic Hole Under Tension

(1) For 0° composites

At Point B

$$\frac{\sigma_\theta}{p} = 1 + n\frac{b}{a} \qquad (79)$$

For B/Ep,

$$\frac{\sigma_\theta}{p} = 1 + 5.98\frac{b}{a} \qquad (80)$$

For Gr/Ep

$$\frac{\sigma_\theta}{p} = 1 + \qquad \frac{b}{a} \qquad (81)$$

(2) For 90° composite

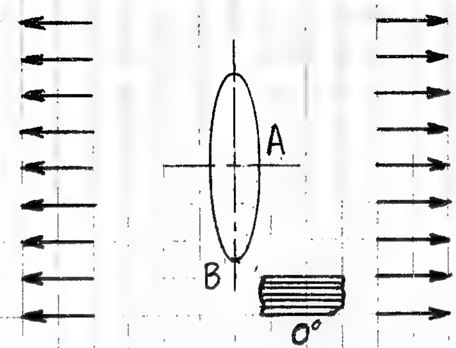


Figure 36 Elliptical opening transverse to the fibers.

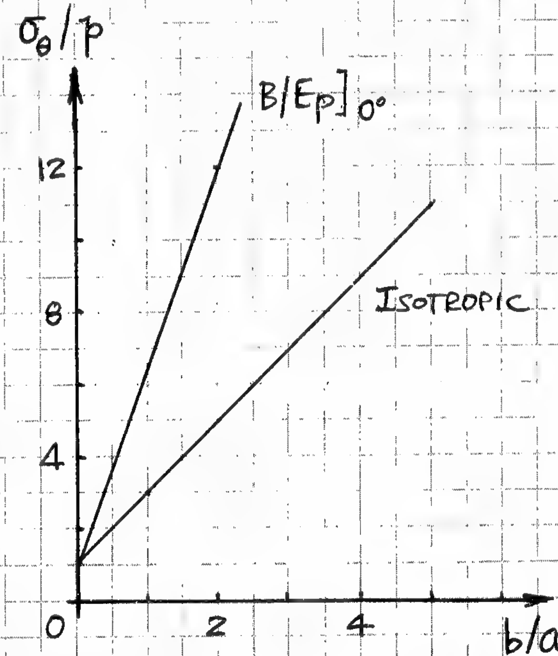


Figure 37 Stress concentration factor for various aspect ratio.

4. TRANSFORMATION RELATIONS FOR COMPLIANCE

a. Formulas

TABLE 17 TRANSFORMATION OF COMPLIANCE

	I_1	I_2	R_1	R_2
S'_{11}	1	1	$-\cos 2(\theta - \delta_1)$	$-\cos 4(\theta - \delta_2)$
S'_{22}	1	1	$\cos 2(\theta - \delta_1)$	$-\cos 4(\theta - \delta_2)$
S'_{12}	1	-1	0	$\cos 4(\theta - \delta_2)$
S'_{66}	0	4	0	$4 \cos 4(\theta - \delta_2)$
S'_{16}	0	0	$\sin 2(\theta - \delta_1)$	$2 \sin 4(\theta - \delta_2)$
S'_{26}	0	0	$\sin 2(\theta - \delta_1)$	$-2 \sin 4(\theta - \delta_2)$

$$I_1 = \frac{1}{4}(S_{11} + S_{22} + 2S_{12}) \tag{82}$$

$$I_2 = \frac{1}{8}(S_{11} + S_{22} - 2S_{12} + S_{66}) \tag{83}$$

$$R_1 = \frac{1}{2}\sqrt{(-S_{11} + S_{22})^2 + (S_{16} + S_{26})^2} \tag{84}$$

$$R_2 = \frac{1}{8}\sqrt{(S_{11} + S_{22} - 2S_{12} - S_{66})^2 + 4(S_{26} - S_{16})^2} \tag{85}$$

$$\tan 2\delta_1 = \frac{S_{16} + S_{26}}{S_{11} - S_{22}} \tag{86}$$

$$\tan 4\delta_2 = \frac{2(S_{16} - S_{26})}{S_{11} + S_{22} - 2S_{12} - S_{66}} \tag{87}$$

For orthotropic material

$$\delta_1 = \delta_2 \tag{88}$$

For anisotropic material

$$\delta_1 \neq \delta_2 \tag{89}$$

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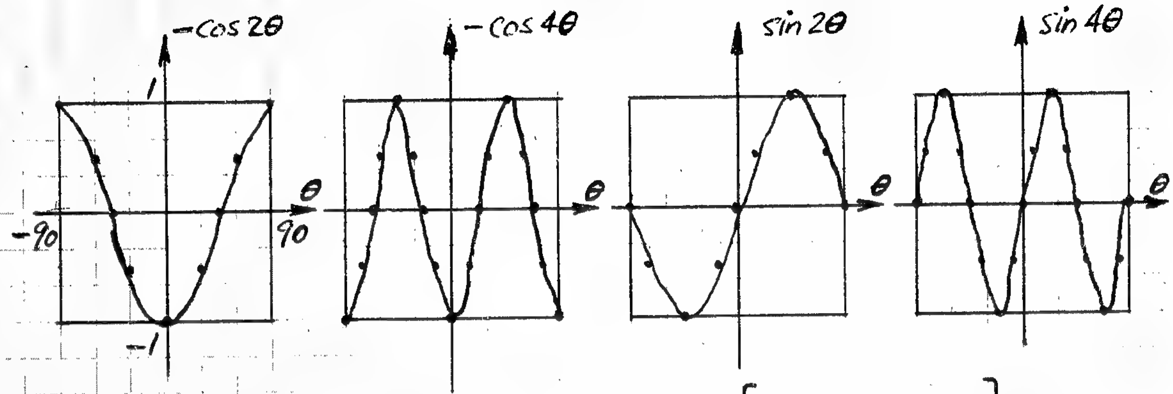
b. Typed Numerical Data

TABLE 18 INVARIANTS FOR COMPLIANCE MATRIX FOR VARIOUS COMPOSITES $(TPa)^{-1}$

Material	I_{1S}	I_{2S}	R_{1S}	R_{2S}	
T-300/5208	24.9	30.6	45.7	4.22	
B/5505	14.2	29.2	24.6	13.9	
Scotchply 1002	33.3	50.2	47.5	10.2	

c. Graphical Illustration

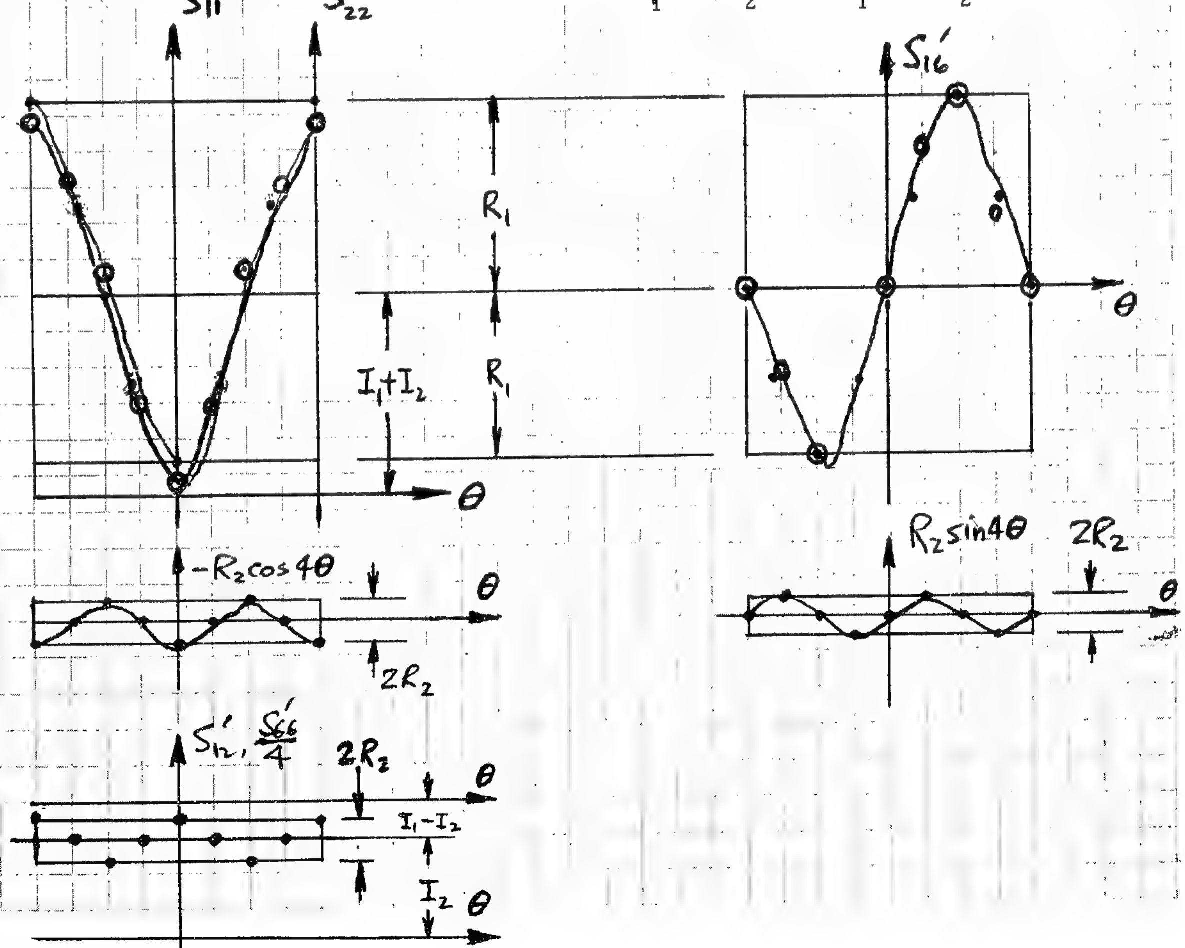
(1) Trigonomatic Relations



(2) Transformed Compliance for T-300/5208,

$$S_{ij} = \begin{bmatrix} 5.5 & -1.5 & 0 \\ -1.5 & 97 & 0 \\ 0 & 0 & 139 \end{bmatrix} (TPa)^{-1}$$

$I_1 = 25$, $I_2 = 30$, $R_1 = 45$, $R_2 = 4$



(3) Variation in Engineering Constants as Coordinates Rotate (GPa)

$$\overline{E}'_{11} = \frac{1}{S'_{11}}, \quad \overline{E}'_{22} = \frac{1}{S'_{22}}, \quad \overline{\nu}'_{12} = -\frac{S'_{12}}{S'_{11}}, \quad \overline{\nu}'_{21} = -\frac{S'_{12}}{S'_{22}}, \quad G'_{12} = \frac{1}{S'_{66}}$$

TABLE 19 TRANSFORMED COMPLIANCE & ENGINEERING CONSTANTS FOR VARIOUS COMPOSITES $(TPa)^{-1}$ AND (GPa)

	T-300 / 5208					
	S_{ij}'			Engineering Constants	S_{ij}'	Engineering Constants
0^o	5.52	-1.54	0	E_{11} = 182		
				E_{22} = 10.3		
		97.1	0	ν_{12} = .28		
				ν_{21} = .016		
			139	G_{12} = 7.2		
15^o	13.7	-3.62	30	E_{11} = 73		
				E_{22} = 10.8		
		92	15.7	ν_{12} = .264		
				ν_{21} = .039		
			130	G_{12} = 7.7		
30^o						
45^o						

5. TRANSFORMATION RELATIONS FOR MODULUS MATRIX

a. Formulas

TABLE 20 TRANSFORMATION OF MODULUS

	I_1	I_2	R_1	R_2
Q'_{11}	1	1	$\cos 2(\theta - \delta_1)$	$\cos 4(\theta - \delta_2)$
Q'_{22}	1	1	$-\cos 2(\theta - \delta_1)$	$\cos 4(\theta - \delta_2)$
Q'_{12}	1	-1	0	$-\cos 4(\theta - \delta_2)$
Q'_{66}	0	1	0	$-\cos 4(\theta - \delta_2)$
Q_{16}	0	0	$-\frac{1}{2}\sin 2(\theta - \delta_1)$	$-\sin 4(\theta - \delta_2)$
Q_{26}	0	0	$-\frac{1}{2}\sin 2(\theta - \delta_1)$	$\sin 4(\theta - \delta_2)$

$$I_1 = \frac{1}{4}(Q_{11} + Q_{22} + 2Q_{12}) \tag{90}$$

$$I_2 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66}) \tag{91}$$

$$R_1 = \frac{1}{2}\sqrt{(-Q_{11} + Q_{22})^2 + 4(Q_{16} + Q_{26})^2} \tag{92}$$

$$R_2 = \frac{1}{8}\sqrt{(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})^2 + 16(Q_{16} - Q_{26})^2} \tag{93}$$

$$\tan 2\delta_1 = -\frac{2(Q_{16} + Q_{26})}{Q_{11} - Q_{22}} \tag{94}$$

$$\tan 4\delta_2 = -\frac{4(Q_{16} - Q_{26})}{Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}} \tag{95}$$

$\delta_1 = \delta_2$ for orthotropic materials

$\delta_1 \neq \delta_2$ for anisotropic materials

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b. Typical Numerical Data

TABLE 21 INVARIANTS FOR MODULUS MATRIX FOR VARIOUS COMPOSITES (GPa)

Material	I_{1Q}	I_{2Q}	R_{1Q}	R_{2Q}	
T-300/5208	49.5	26.9	85.7	19.7	
B/5505	58.0	29.8	93.2	24.0	
Scotchply 1002	13.0	7.47	15.4	3.33	

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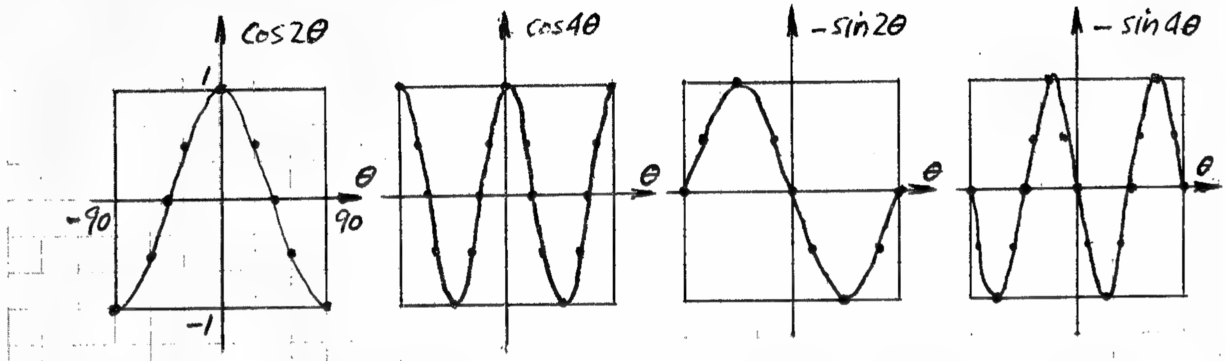
c. Graphical Illustration

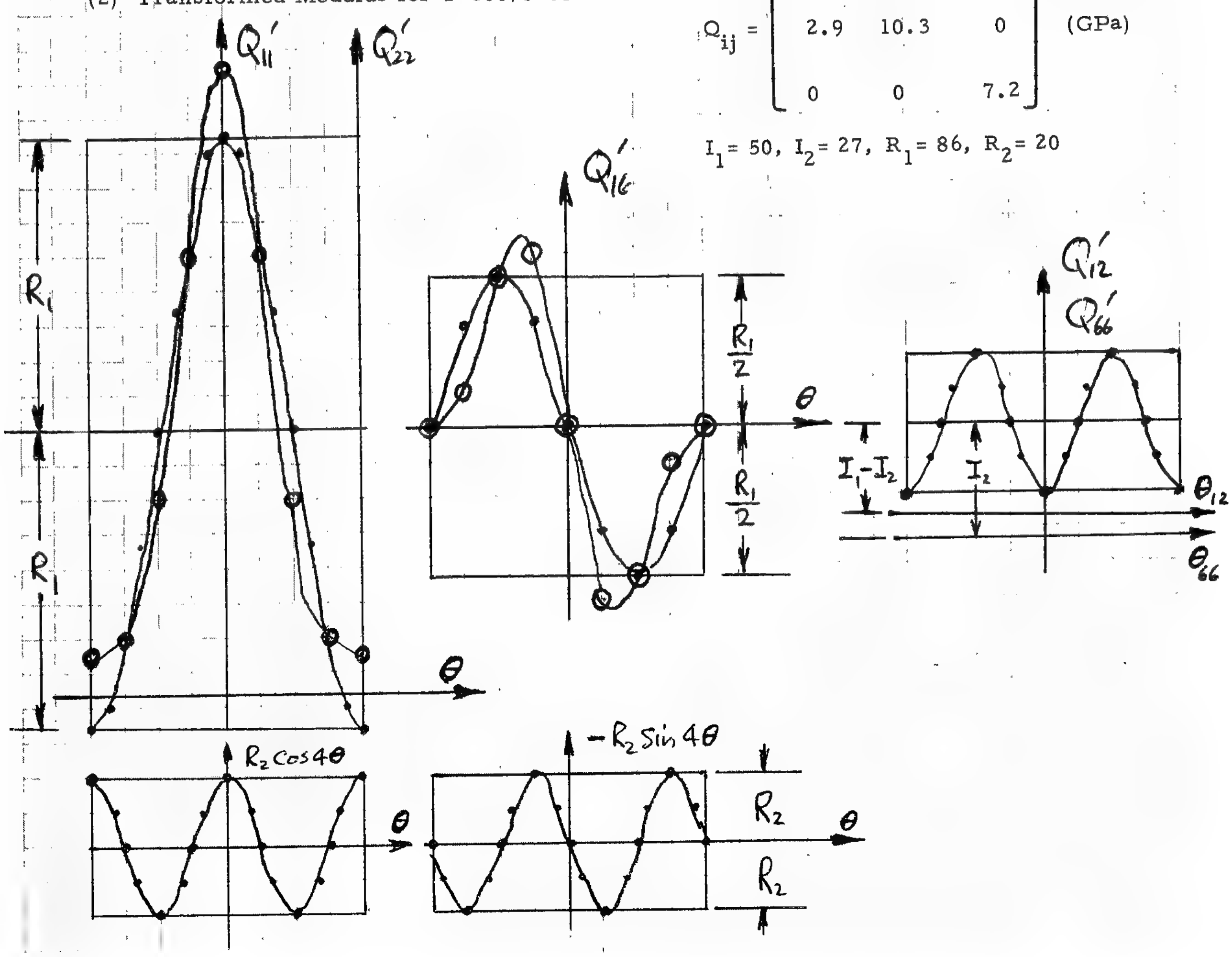
(1) Trigonometric Relations



(2) Transformed Modulus for T-300/5208

$$Q_{ij} = \begin{bmatrix} 182 & 2.9 & 0 \\ 2.9 & 10.3 & 0 \\ 0 & 0 & 7.2 \end{bmatrix} \text{ (GPa)}$$

$I_1 = 50$, $I_2 = 27$, $R_1 = 86$, $R_2 = 20$



d. Typical Data for Transformed Modulus for Various Composites

TABLE 22 TRANSFORMED MODULUS FOR VARIOUS COMPOSITES (GPa)

	T-300/5208			
$Q_{ij}^{(0)}$	$\begin{matrix} 182 & 2.9 & 0 \\ & 10.3 & 0 \\ & & 7.2 \end{matrix}$			
$Q_{ij}^{(15)}$	$\begin{matrix} 160 & 13 & -38 \\ & 12 & -4 \\ & & 17 \end{matrix}$			
$Q_{ij}^{(30)}$				
$Q_{ij}^{(45)}$				

6. MODULI FOR RANDOM COMPOSITES

a. Constant Stress or Series Model

$$\bar{e}_i = \sigma_j \int_{-\pi/2}^{\pi/2} S_{ij}\, d\theta = \begin{bmatrix} I_1+I_2 & I_1-I_2 & 0 \\ & I_1+I_2 & 0 \\ & & 4I_2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{96}$$

The invariants are those for S_{ij} in Table 18.

$$E_s = \frac{1}{I_1+I_2} \tag{97}$$

$$\nu_s = -\frac{I_1-I_2}{I_1+I_2} \tag{98}$$

$$G_s = \frac{1}{4I_2} \tag{99}$$

b. Constant Strain or Parallel Model

$$\bar{\sigma}_i = e_j \int_{-\pi/2}^{\pi/2} Q_{ij}\, d\theta = \begin{bmatrix} I_1+I_2 & I_1-I_2 & 0 \\ & I_1+I_2 & 0 \\ & & I_2 \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_6 \end{Bmatrix} \tag{100}$$

The invariants are those for Q_{ij} in Table 21.

$$\nu_p = \frac{I_1-I_2}{I_1+I_2} \tag{101}$$

$$E_p = (1-\bar{\nu}^2)(I_1+I_2) \tag{102}$$

$$G_p = I_2 \tag{103}$$

c. Isotropic Constants for Various Random Composites

TABLE 23 PREDICTED ELASTIC CONSTANTS FOR RANDOM COMPOSITES (GPa)

Material	E_s	G_s	E_p	G_p	
T-300/5208	18.	8.17	69.7	26.9	
B/5505	23.0	8.56	78.7	29.8	
Scotchply 1002	12.0	4.98	19.0	15.4	

SECTION IV

MICROMECHANICS

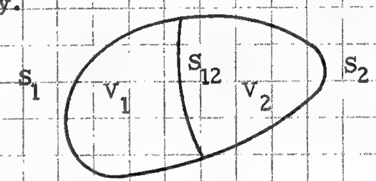
1. ELASTIC PROPERTIES

a. Definitions

Average strains depend on the boundary displacement only.

$$\bar{e}_{ij} = \frac{1}{V}\int_V e_{ij}\,dV = \frac{1}{2V}\int_V (u_{i,j} + u_{j,i})\,dV$$

$$= \frac{1}{2V}\int_S (u_i n_j + u_j n_i)\,dS \qquad (104)$$



$S = S_1 + S_2$

$V = V_1 + V_2$

Figure 38 A composite body. Displacement continuity and equilibrium are maintained at the interface.

Average stresses depend on the boundary tractions only.

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_V \sigma_{ij}\,dV = \frac{1}{V}\int_V (\sigma_{ik} x_j)_{,k}\,dV$$

$$= \frac{1}{2V}\int_S (T_i x_j + T_j x_i)\,dS \qquad (105)$$

Virtual work

$$\overline{\sigma_{ij} e_{ij}} = \frac{1}{V}\int_V \sigma_{ij} e_{ij}\,dV = \frac{1}{V}\int_S T_i u_i\,dS \qquad (106)$$

σ_{ij}: statically admissible, u_i: kinematically admissible

Divergence theorem

$$\int_S f_i n_j\,dS = \int_V f_{i,j}\,dV \qquad (107)$$

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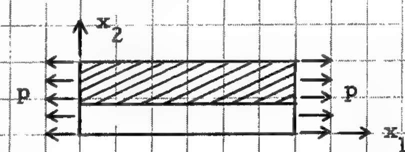
(1) Sample Problems

(a) A composite bar of length ℓ is given a displacement $\delta\ell$ at one end while the other end is held fixed. What is the average strain $\overline{e}_{11}$?

(b) A composite bar is subjected to the pressure p at both ends. What is the average stress $\overline{\sigma}_{11}$?



b. Homogeneous Boundary Conditions

If $u_i = e^o_{ij} x_j$ on S, then $\bar{e}_{ij} = e^o_{ij}$. (108)

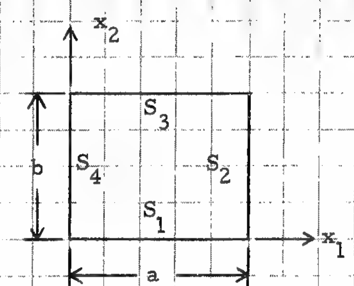
If $T_i = \sigma^o_{ij} x_j$ on S, then $\bar{\sigma}_{ij} = \sigma^o_{ij}$. (109)

If either of the above, then $\overline{\sigma_{ij} e_{ij}} = \bar{\sigma}_{ij} \bar{e}_{ij}$. (110)

The above reductions follow from the definitions and the divergence theorem.

(1) Sample Problem

(a) Prescribe the homogeneous boundary conditions on a two-dimensional, rectangular body.



c. Effective Elastic Constants

Displacement B.C. $u_i = \overset{o}{e}_{ij} x_j$ on S.

$$u_i(x) = u_{ik\ell}(x) \overset{o}{e}_{k\ell} \quad \text{in V} \tag{111}$$

$$e_{ij}(x) = e_{ijk\ell} \overset{o}{e}_{k\ell} , \quad e_{ijk\ell} = \frac{1}{2}(u_{ik\ell,j} + u_{jk\ell,i}) \tag{112}$$

$$\sigma_{ij}(x) = C_{ijk\ell}(x) e_{k\ell}(x) \tag{113}$$

$$\therefore \overline{\sigma}_{ij} = C^*_{ijk\ell} \overline{e}_{k\ell} , \quad C^*_{ijk\ell} = \frac{1}{V}\int_V C_{ijmn}(x) e_{mnk\ell}(x) dV \tag{114}$$

Traction B.C. $T_i = \overset{o}{\sigma}_{ij} n_j$

$$\overline{e}_{ij} = S^*_{ijk\ell} \overline{\sigma}_{k\ell} , \quad S^*_{ijk\ell} = \frac{1}{V}\int_V S_{ijmn}(x) \sigma_{mnk\ell}(x) dV \tag{115}$$

$$\sigma_{ij}(x) = \sigma_{ijk\ell}(x) \overset{o}{\sigma}_{k\ell} \tag{116}$$

Strain energy

$$\overline{W}^e = \frac{1}{2} C^*_{ijk\ell} \overset{o}{e}_{ij} \overset{o}{e}_{k\ell} = \frac{1}{2} C^*_{ijk\ell} \overline{e}_{ij} \overline{e}_{k\ell} \quad : \text{Displacement B.C.} \tag{117}$$

$$\overline{W}^\sigma = \frac{1}{2} S^*_{ijk\ell} \overset{o}{\sigma}_{ij} \overset{o}{\sigma}_{k\ell} = \frac{1}{2} S^*_{ijk\ell} \overline{\sigma}_{ij} \overline{\sigma}_{k\ell} \quad : \text{Traction B.C.} \tag{118}$$

Statistically homogeneous body

$$C^*_{ijk\ell} S^*_{k\ell mn} = I_{ijmn}, \quad I_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{119}$$

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d. Prediction Based on Averages

Determine : $\bar{e}_i = S^*_{ij}\bar{\sigma}_j$ (6 equations) (120)

Unknowns : $\bar{\sigma}_i$, $\bar{e}_i$, $\bar{\sigma}_{fi}$, $\bar{e}_{fi}$, $\bar{\sigma}_{mi}$, $\bar{e}_{mi}$ (36) (121)

Available Equations:

Const. Eqns. $\bar{e}_{fi} = S_{fij}\bar{\sigma}_{fj}$ $\bar{e}_{mi} = S_{mij}\bar{\sigma}_{mj}$ (12 equations) (122)

Definitions $\bar{e}_i = v_f\bar{e}_{fi} + v_m\bar{e}_{mi}$, $\bar{\sigma}_i = v_f\bar{\sigma}_{fi} + v_m\bar{\sigma}_{mi}$ (12 equations) (123)

No. of additional equations required : 6

To be provided from the equilibrium and compatibility conditions.

Suppose

$$\bar{\sigma}_{fi} = B_{fij}\bar{\sigma}_j \tag{124}$$

Then

$$S^*_{ij} = v_f S_{fik} B_{fkj} + S_{mik}(\delta_{kj} - v_f B_{fkj}) \tag{125}$$

e. Approximation for Unidirectional Laminae

For equilibrium

$$\bar{\sigma}_{mi} = \eta_i \bar{\sigma}_{fi} , \quad i = 2\text{-}6 . \tag{126}$$

For compatibility

$$\bar{\epsilon}_{f1} = \eta_1 \bar{\epsilon}_{m1} \tag{127}$$

$$E_1^* = \frac{1}{\eta_1 v_f + v_m} (\eta_1 v_f E_f + v_m E_m) \tag{128}$$

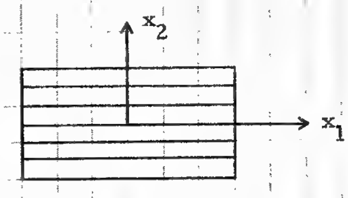


Figure 39 Unidirectional composite and the reference coordinate system.

$$\nu_{12}^* = \frac{1}{\eta_1 v_f + v_m} (\eta_1 v_f \nu_f + v_m \nu_m) \tag{129}$$

$$\frac{1}{E_a^*} = \left(\frac{v_f}{E_f} + \frac{\eta_a v_m}{E_m} \right) \frac{1}{v_f + \eta_a v_m} - v_f v_m \frac{(\eta_1 \eta_a E_f \nu_m - E_m \nu_f)(\nu_m/E_m - \nu_f/E_f)}{(\eta_1 v_f E_f + v_m E_m)(v_f + \eta_a v_m)} \tag{130}$$

$$\nu_{a1}^* = \nu_{1a}^* E_a^* / E_1^* \tag{131}$$

$$\frac{1}{G_{cc}^*} = \left(\frac{v_f}{G_f} + \frac{\eta_c v_m}{G_m} \right) \frac{1}{v_f + \eta_c v_m} \tag{132}$$

$$\nu_{ab}^* = \frac{E_a^*}{v_f + \eta_a v_m} \left(v_f \frac{1 + \nu_f}{E_f} + \eta_a v_m \frac{1 + \nu_m}{E_m} \right) - 1 \tag{133}$$

$a, b = 2, 3 \qquad c = 4, 5, 6$

Note that

$$\frac{\nu_{23}^*}{E_2^*} \neq \frac{\nu_{32}^*}{E_3^*} \quad \text{unless } \eta_2 = \eta_3 . \tag{134}$$

f. Transverse Isotropy

$$E_2^* = E_3^* \longrightarrow \eta_2 = \eta_3 \tag{135}$$

$$-\frac{\nu_{23}^*}{E_2^*} = \frac{1}{E_2^*} - \frac{1}{2G_{44}^*} \longrightarrow \eta_2 = \eta_4 \tag{136}$$

$$G_{44}^* = G_{55}^* \longrightarrow \eta_4 = \eta_5 \tag{137}$$

Independent η's : η_1, η_2, η_6

Independent elastic moduli :

$$E_L = E_1^* \tag{138}$$

$$E_T = E_2^* = E_3^* \tag{139}$$

$$\nu_{LT} = \nu_{TL} E_L / E_T = \nu_{12}^* \tag{140}$$

$$G_{LT} = G_{66}^* \tag{141}$$

$$\nu_{TT} = \nu_{23}^* = \nu_{32}^* \tag{142}$$

$G_{TT} (= G_{44}^* = G_{55}^*)$ is determined from (143)

$$G_{TT} = \frac{E_T}{2(1+\nu_{TT})} \tag{144}$$

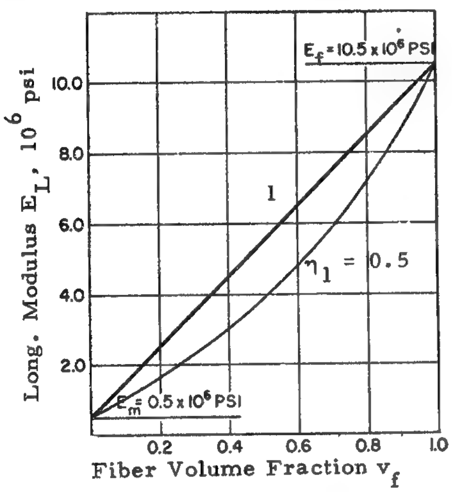


Figure 40 Longitudinal modulus can be predicted by Eq. (128) with $\eta_1 = 1$. Constituents are E-glass fibers (E_f = 72.4 GPa, ν_f = 0.20) and epoxy matrix (E_m = 3.45 GPa, ν_m = 0.35).

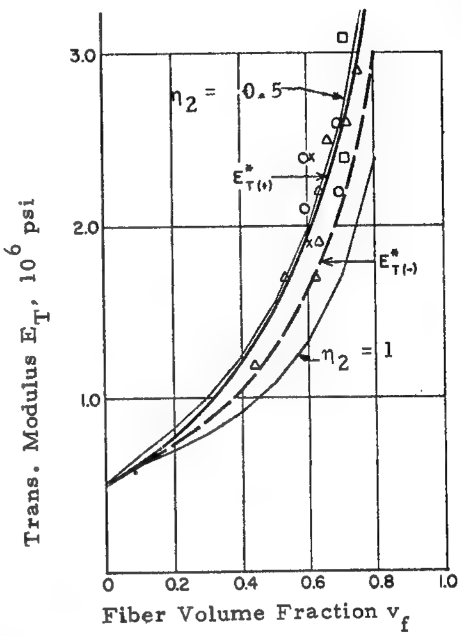


Figure 41 Prediction of transverse modulus by Eq. (130) with $\eta_2 = 0.5$, $\eta_1 = 1$, is comparable to the more rigorous solutions. Constituents are E-glass fibers (E_f = 73.1 GPa, ν_f = 0.22) and epoxy matrix (E_m = 3.45 GPa, ν_m = 0.35). [2, 3]

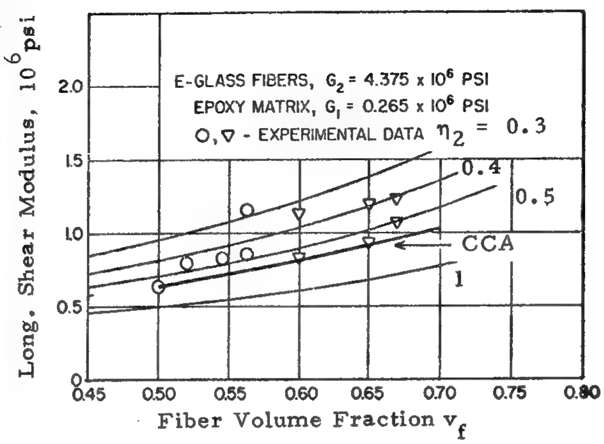


Figure 42 Longitudinal shear modulus increases with decreasing η_2 when predicted by Eq. (132). Good agreement is seen when η_2 = 0.4 - 0.5. [2, 4]

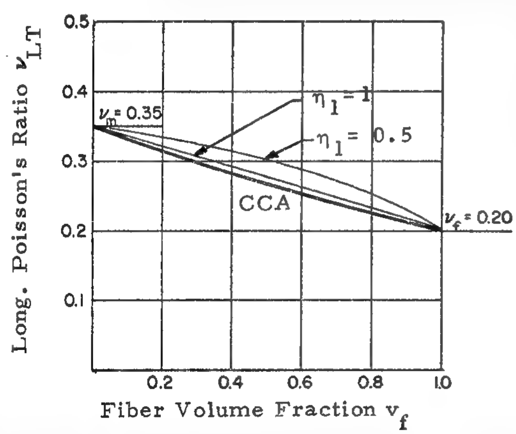


Figure 43 Longitudinal Poisson's ratio can be predicted by Eq. (129) with η_1 = 1. Constituent properties are the same as those in Figure 40. [2]

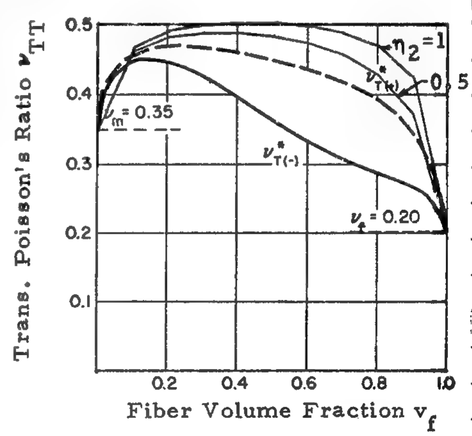


Figure 44 Transverse Poisson's ratio predicted by Eq. (133) with η_2 = 0.5, η_1 = 1, is higher than the upper bound from a more rigorous solution. [2]

g. Approximations for Particulate Composite

Assume

$$\overline{\sigma}_{mi} = \eta_1 \overline{\sigma}_{fi}, \quad i = 1 - 3; \quad \overline{\sigma}_{mi} = \eta_2 \overline{\sigma}_{fi}, \quad i = 4 - 6. \tag{145}$$

$$\frac{1}{E^*} = \left(\frac{v_f}{E_f} + \frac{\eta v_m}{E_m}\right) \frac{1}{v_f + \eta v_m} \tag{146}$$

$$\nu^* = \frac{v_f \nu_f E_m + \eta v_m \nu_m E_f}{v_f E_m + \eta v_m E_f} \tag{147}$$

Isotropy requires that

$$\eta_1 = \eta_2 = \eta$$

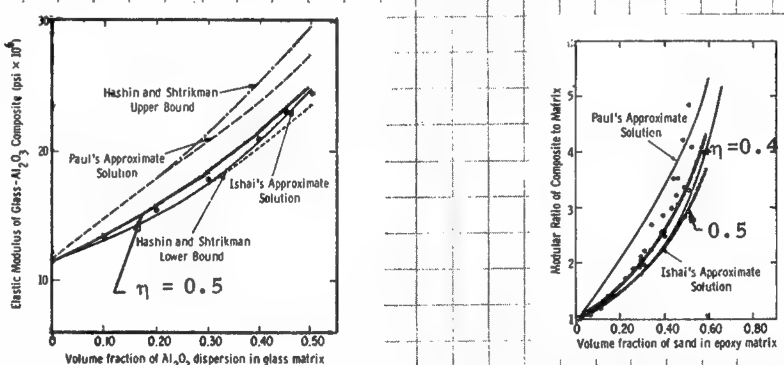


Figure 45 Young's modulus of particulate composite can be predicted by Eq. (146) with η = 0.4-0.5 ; (a) Al_2O_3/glass ($E_f \approx 396$ GPa, $E_m \approx 79.3$ GPa); (b) Sand/epoxy ($E_f \approx 124$ GPa , $E_m \approx 3.45$ GPa). [5]

h. Thermal Expansion and Swelling Coefficients

Hygro-thermo-elastic relations for homogeneous constituent materials

$$e_i = S_{ij}\sigma_j + \alpha_i \theta \quad \text{or} \tag{148}$$

$$\sigma_i = C_{ij} e_j + \beta_i \theta \quad ,$$

where θ is the temperature difference (T) or the moisture concentration c . If both, $\alpha_i \theta$ should be replaced by $(\alpha_i^T T + \alpha_i^H c)$.

Let e_i^σ and e_i^θ be the strains resulting respectively from the homogeneous stress B.C. with $\theta = 0$ and the stress free B.C. with $\theta = \theta^o$.

Then

$$\overline{e}_i^\sigma = S_{ij}^* \sigma_j^o = S_{ij}^* \overline{\sigma}_j , \quad \overline{e_i^\theta} = \alpha_i^* \theta^o = \alpha_i^* \overline{\theta} \tag{149}$$

Therefore

$$\overline{e}_i = S_{ij}^* \overline{\sigma}_j + \alpha_i^* \overline{\theta} \quad . \tag{150}$$

Similarly

$$\overline{\sigma}_i = C_{ij}^* \overline{e}_j + \beta_i^* \overline{\theta} \quad . \tag{151}$$

(1) Sample Problem [2]

(a) Show that

$$\alpha_i^* = \alpha_{mi} + (\alpha_{fj} - \alpha_{mj}) S^{-1}_{fmjk} (S^*_{ki} - S_{mki}) \tag{152}$$

where

$$S_{fmij} = S_{fij} - S_{mij} \tag{153}$$

Proof

From the virtual work theorem

$$\overline{\sigma_i^\sigma e_i^\theta} = \bar{\sigma}_i \overline{e_i^\theta} = \bar{\sigma}_i \alpha_i^* \bar{\theta} \tag{154}$$

But

$$\overline{\sigma_i^\sigma e_i^\theta} = \overline{S_{ij} \sigma_i^\sigma \sigma_j^\theta} + \overline{\sigma_i^\sigma \alpha_i}\, \bar{\theta} \tag{155}$$

$$= \overline{e_i^\sigma \sigma_j^\theta} + \overline{\sigma_i^\sigma \alpha_i}\, \bar{\theta} = \overline{\sigma_i^\sigma \alpha_i}\, \bar{\theta} \tag{156}$$

Therefore

$$\bar{\sigma}_i \alpha_i^* = v_m \alpha_{mi} \overline{\sigma^\sigma_{mi}} + v_f \alpha_{fi} \overline{\sigma^\sigma_{fi}} \tag{157}$$

Let

$$\overline{\sigma^\sigma_{fi}} = B_{fij} \bar{\sigma}_j \text{ . Then,} \tag{158}$$

$$S^*_{ik} = S_{mik} + v_f (S_{fij} - S_{mij}) B_{fjk} \quad \text{or} \tag{159}$$

$$v_f B_{fij} = S^{-1}_{fmik} (S^*_{kj} - S_{mkj}) \tag{160}$$

Noting that $v_m \overline{\sigma^\sigma_{mi}} = \bar{\sigma}_i - v_f \overline{\sigma^\sigma_{fi}}$, we obtain

$$\bar{\sigma}_i \alpha_i^* = \alpha_{mi} \bar{\sigma}_i + (\alpha_{fi} - \alpha_{mi}) S^{-1}_{fmik} (S^*_{kj} - S_{mkj}) \bar{\sigma}_j \text{ .} \tag{161}$$

Since $\bar{\sigma}_i$ is arbitrary, the proof follows.

i. Approximations for α_i^*

Following the same procedure as in Section e, we obtain

$$\alpha_L = \alpha_1^* = (v_f E_f \alpha_f + v_m E_m \alpha_m)/(\eta_1 v_f E_f + v_m E_m) \tag{162}$$

$$\alpha_T = \alpha_2^* = v_f \alpha_f + v_m \alpha_m + (v_f \alpha_f \nu_f + v_m \alpha_m \nu_m) - (\eta_1 v_f \nu_f + v_m \nu_m)\alpha_1^* \tag{163}$$

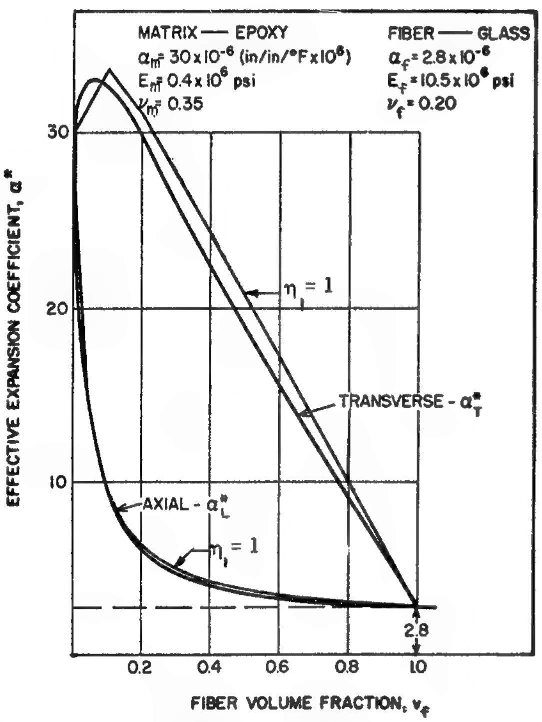


Figure 46 Thermal expansion coefficents predicted by Eqs. (162-163) with $\eta_1 = 1$ are comparable to the more rigorous solutions. [2]

TABLE 24 TYPICAL COMPOSITE THERMAL EXPANSION COEFFICIENTS

COMPOSITE	CONSTITUENTS	v_f	LONG EXP. m/m/°K	TRANS. EXP. m/m/°K
B/Ep	B_4/5505	0.5	4.32	22.1
B/PI	B_4/WRD 9371	0.49	4.90	28.4
Gr/Ep	Mod II/5206	0.55	-0.23	34.0
Gr/Ep	HMS/3002M	0.48	-0.23	33.5
Gr/Ep	T300/5208	0.70	0.01	12.5
Gr/Ep	Mod I/ERLA 4289	0.51	-1.10	31.5
Gr/Ep	Mod I/ERLA 4617	0.45	-0.90	33.3
Gr/PI	Mod I/WRD 9371	0.45	0.	25.3
Gl/Ep	S-Glass/1009	0.72	3.8	16.7
Gl/Ep	Scotchply 1002	0.45	4.16	15.5

a. Microstresses Under Longitudinal Tension

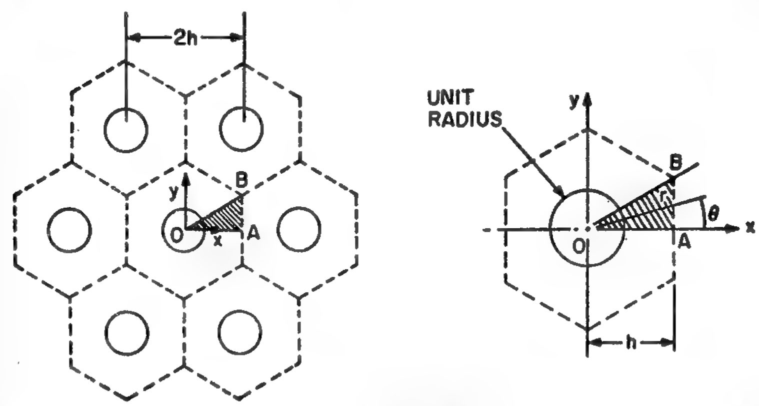


Figure 47 Hexagonal array of fibers and the coordinate system chosen.

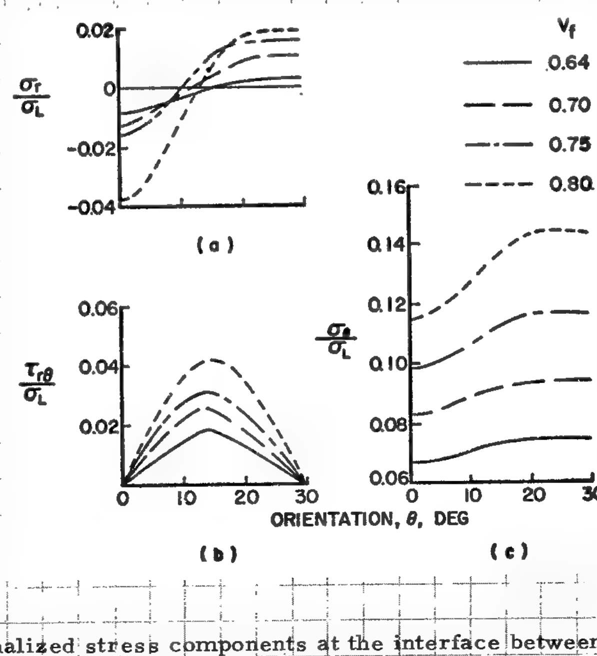


Figure 48 Normalized stress components at the interface between the fiber and matrix under longitudinal loading (E_f=414 GPa, E_m=2.62 GPa). The radial stress becomes tensile at the resin-rich area (in the vicinity of $\theta = 30^o$) even under longitudinal tension. [6]

b. Stress Concentration Under Transverse Tension

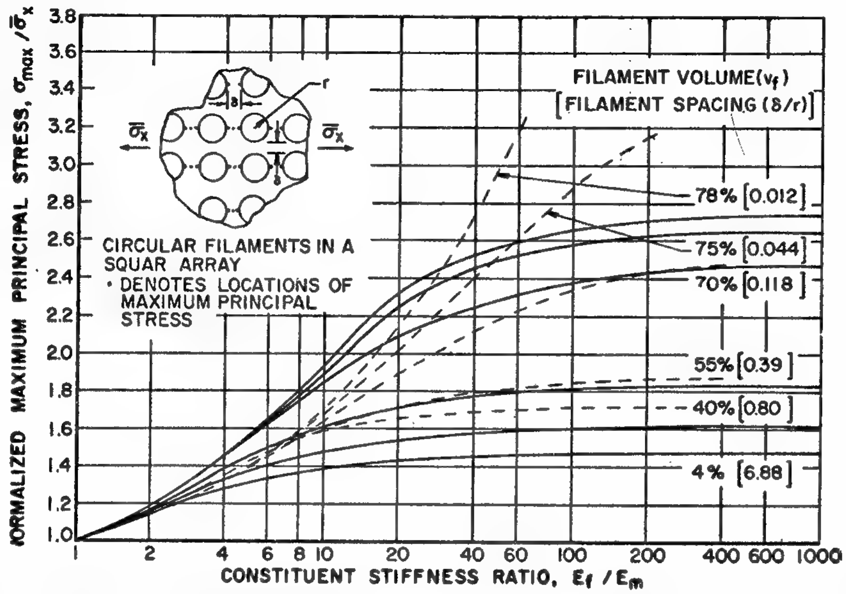


Figure 49 Stress concentration under transverse tension. Solid and broken lines represent the stress concentration factors at the interface and in the matrix, respectively. [7, 8]

c. Stress Concentration Under Longitudinal Shear

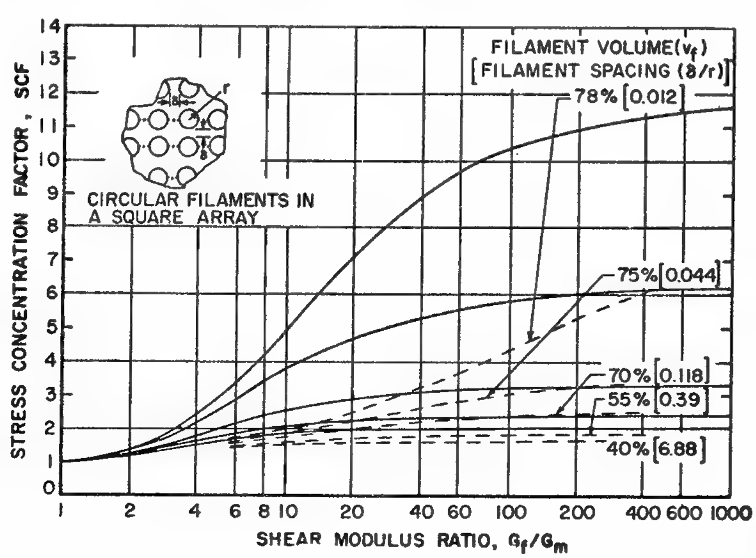


Figure 50 Stress concentration under longitudinal shear. Solid and broken lines represent the stress concentration factors at the interface and in the matrix, respectively. [7, 8]

d. Residual Stresses After Curing

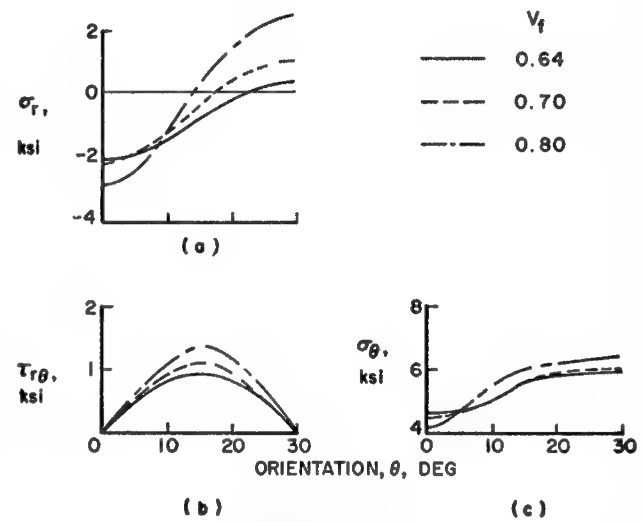


Figure 51 Residual stresses at the interface in a composite with hexagonal array of fibers. Constituent properties are the same as those in Figure 48 , and the stresses correspond to the unconstrained matrix shrinkage strain of 1 cm/m. [6]

TABLE 25 TOTAL STRESSES UNDER LONGITUDINAL TENSION (cf. [9])

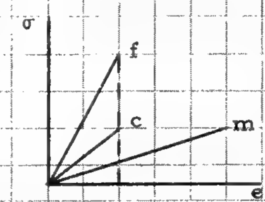
Fiber Volume Content V_f	Modulus Ratio E_f / E_m	Radial Stress, σ_r (ksi) $\theta = 0^\circ$	Radial Stress, σ_r (ksi) $\theta = 30^\circ$	Hoop Stress, σ_θ (ksi) $\theta = 30^\circ$	Maximum Shear Stress, $\tau_{r\theta}$ (ksi)
Curing Stresses (Resin shrinkage = 0.6%)					
0.64	150	-1.2	0.3	3.0	0.6
0.70	150	-1.2	0.6	3.6	0.6
0.64	26	-1.5	0.3	3.0	0.6
0.70	26	-1.8	0.6	3.6	0.6
Stresses Due to Longitudinal Tension of 100 ksi					
0.64	150	-0.7	0.4	7.5	1.5
0.70	150	-1.2	1.0	9.0	2.5
0.64	26	-1.0	1.0	7.5	2.0
0.70	26	-1.3	1.3	9.0	2.5
Combined Stresses					
0.64	150	-1.7	0.7	10.5	2.1
0.70	150	-2.4	1.6	12.6	3.1
0.64	26	-2.5	1.3	10.5	2.6
0.70	26	-3.1	1.9	12.6	3.1

3. STRENGTH

a. Longitudinal Tensile Strength

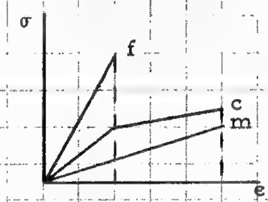
Brittle matrix

$$X = [v_f + (1-v_f)E_m/E_f]X_f \qquad (164)$$



Ductile matrix (tough)

$$X = v_f X_f + v_m X_m \qquad (165)$$



Influencing factors

Fiber damage - fiber surface treatment (Gr fibers), heat treatment (B/Al)

Strength of B_4/Al-6061 (v_f = 0.25)

Heat treatment	No treatment
98 MPa	104 MPa

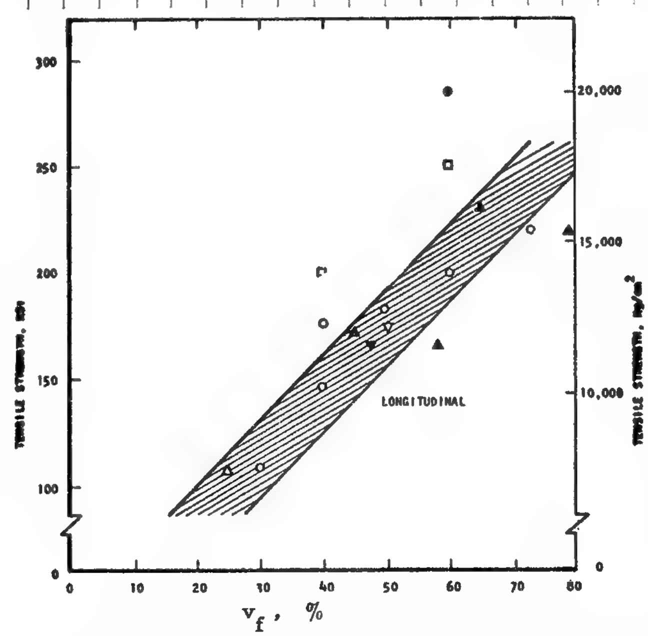


Figure 52 Longitudinal tensile strengths of B_4/Al-6061 and BSC/Al-6061. [10]

TABLE 26 TYPICAL FIBER STRENGTHS

MATERIAL	DIAMETER μm	TENSILE STRENGTH MPa
Kevlar	11.7	2760
Graphite (Gr)	7.6	
MODMOR II HTS		2760
MODMOR I HMS		1850
T300 AS		2240
Glass (Gl)		
E-Glass	5-10	3450
S-Glass	2.5	4480
Boron (B)	100, 140	3450
Borsic (BSC)	140	3010
Steel	13	4140
Tungsten (W)	13	4000
Beryllium (Be)	127	1280

TABLE 27 TYPICAL MATRIX STRENGTHS

MATERIAL	TENSILE STRENGTH MPa	COMPRESSIVE STRENGTH MPa	SHEAR STRENGTH MPa
Epoxy (Ep)			
Narmco 2387	29	159	10
Narmco 5505	59	128	
Epon 828	72	150	83
1004	82	207	134
ERLA 4289	34	93	
ERLA 4617	132	226	
Polyester	72		
Aluminum (Al)			
2024 - T3	427	221	255
6061 - T6	290	241	186
6061 - 0	131	-	83
Titanium (T_i)			
6Al-4V	958	951	558
Pure	552	483	290

TABLE 28 TYPICAL COMPOSITE STRENGTHS [11, 12]

COMPOSITE	CONSTITUENTS	v_f	LONG. TENS. MPa	LONG. COMP. MPa	TRANS. TENS. MPa	TRANS. COMP. MPa	SHEAR MPa
B/Ep	B_4/5505	0.5	1260	2500	61	202	67
B/PI	B_4/WRD 9371	0.49	1040	1090	11	63	26
Gr/Ep	Mod II/5206	0.55	1110	970	36	170	63
Gr/Ep	HMS/3002M	0.48	680	690	16	186	72
Gr/Ep	T300/5208	0.70	1500	1500	40	246	68
Gr/Ep	Mod I/ERLA 4617	0.45	840	880	42	197	62
Gr/PI	Mod I/WRD 9371	0.45	807	652	15	71	22
Gl/Ep	S-Glass/1009	0.72	1290	822	46	174	45
B/Al	B_4/Al-6061-F	0.5	1110	1480	103	159	103
BSC/Ti	BSC/Ti-6Al-4V	0.5	1310	-	441	-	483
Gr/Ep	Mod I/4289	0.5	1120	990	4.2	-	34
Gl/Ep	Scotchply 1002	0.45	1062	610	31.4	118	72

b. Transverse Tensile Strength

Failure occurs when

$K_T \bar{\sigma}_m = X_m$, K_T : effective stress concentration factor

$$\therefore Y = \frac{1}{K_{Tm}} (v_m + v_f/\eta_2) X_m \leq X_m \quad : \text{ matrix failure} \tag{166}$$

$$= \frac{1}{K_{Ti}} (v_m + v_f/\eta_2) X_i \leq X_i \quad : \text{ interface failure} \tag{167}$$

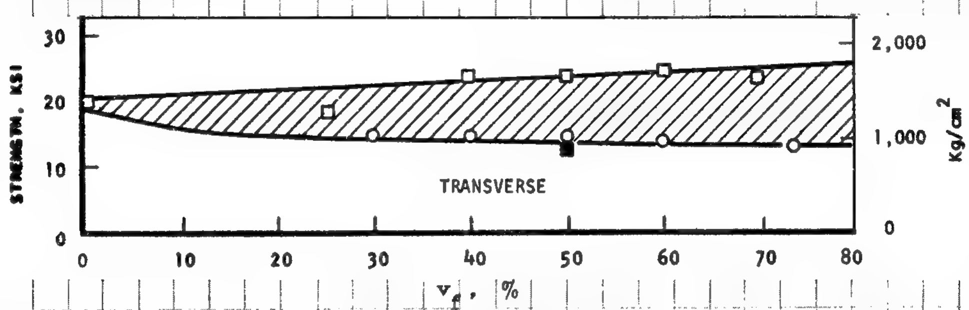


Figure 53 Transverse strengths of B_4/Al-6061 and BSC/Al-6061 [10]

Influencing factors

Fiber splitting decreases Y - B_4/Al

Heat treatment increases X_m and in turn Y .

Voids increase K_T and in turn decrease Y .

Fiber surface treatment increases interface strength and in turn Y .

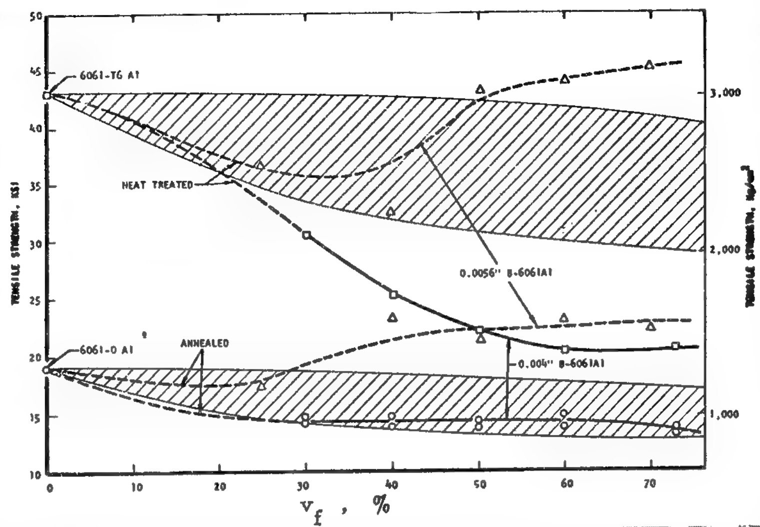


Figure 54 Effect on transverse strength of fiber diameter and matrix condition (B/Al). Failure of B_4/Al-6061 is due to fiber splitting. [10]

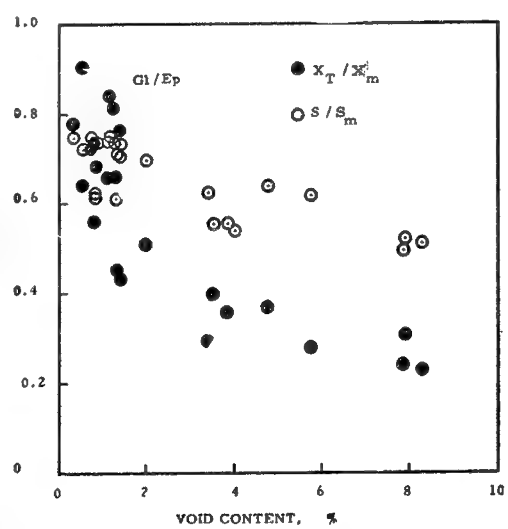


Figure 55 Effect of void content on transverse and shear strengths of Gl/Ep (X_m = 82 MPa, S_m = 134 MPa, v_f = 0.56). [8]

c. Longitudinal Shear Strength

$$S = \frac{1}{K_{Sm}} (v_m + v_f/\eta_6) S_m \leq S_m \quad : \quad \text{matrix failure} \tag{168}$$

$$= \frac{1}{K_{Si}} (v_m + v_f/\eta_6) S_i \leq S_i \quad : \quad \text{interface failure} \tag{169}$$

Influencing factors

Same as those for transverse strength.

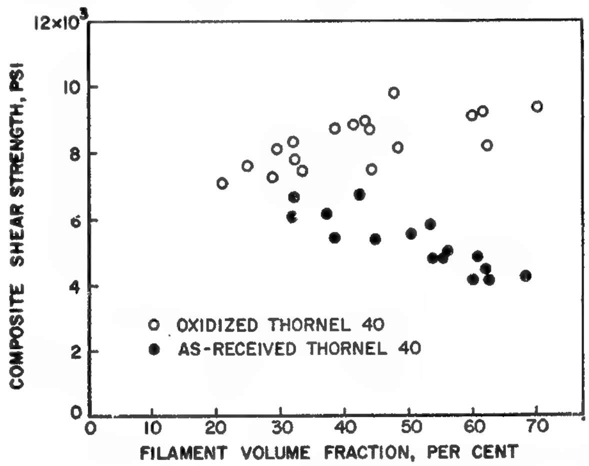


Figure 56 Effect of fiber surface treatment on shear strengths of Gr/Ep. Note that K_{Si} increases more rapidly with v_f than does K_{Sm}. [13]

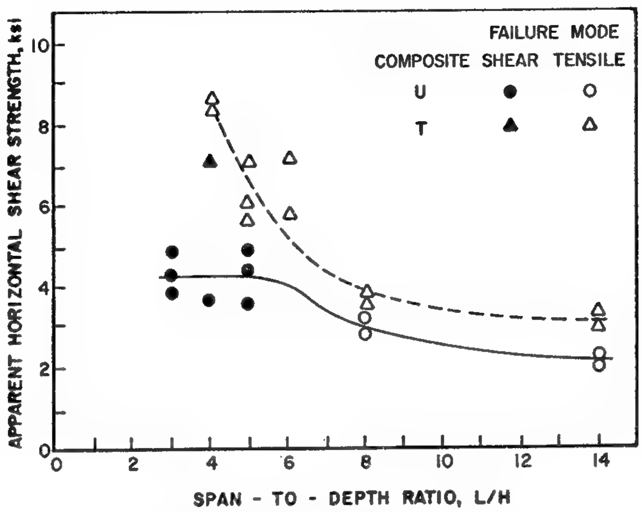


Figure 57 Failure mode in flexure test depends on the shear strength (Gr/Ep, $v_f = 0.6$): U - fiber surface untreated; T - fiber surface treated. [14]

d. Longitudinal Compression Strength

(1) Buckling in Shear Mode

$$X' = \frac{G_m}{1-v_f} + \lambda \frac{\pi^2 v_f}{16} E_f \left(\frac{d_f}{L}\right)^2 \quad \text{from [15]} \tag{170}$$

L : gage length

$$\lambda = \begin{cases} 1 \text{ for simply supported ends} \\ 4 \text{ for fixed ends} \end{cases}$$

(2) Actual Composites

$$\frac{G_m}{1-v_f} \approx \frac{3}{1-v_f} \text{ GPa} \quad : \quad \text{too high !} \tag{171}$$

$$X' \leq v_f X'_f \tag{172}$$

For Gr fibers $X'_f \approx X_f$

$$\frac{X'}{v_f X'_f} \approx 0.36 - 0.84 \quad \text{for some Gr/Ep. [15]} \tag{173}$$

For T300/5208

$X' \approx X$.

Influencing factors

Debond
Bowed fibers
Misaligned fibers
Voids

e. Transverse Compressive Strength

$$Y' \leq X'_m \tag{174}$$

Compare the transverse strengths in Table 28.

4. SYNERGISTIC EFFECTS OF MATRIX AND FIBER STRENGTH SCATTER

a. Role of Matrix

Minimizes interaction among fiber breakages - beneficial.

Induces stress concentration and hence possible fracture - deleterious.

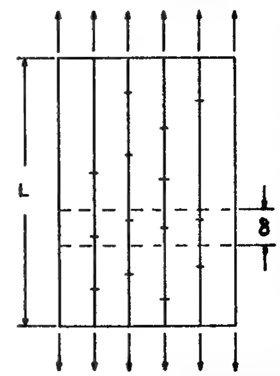


Figure 58 Fiber breakages in composite.

b. Bundle Strength

Fiber strength in stress

$$R_f(X_f) = \exp[-L(X_f/X_{fo})^{\alpha}], \quad \bar{X}_f = X_{fo}L^{-1/\alpha}\Gamma(1 + 1/\alpha) \tag{175}$$

Linear stress-strain relation

$$\sigma = E_f \epsilon \tag{176}$$

Fiber strength in strain

$$R_f(Y_f) = \exp[-L(Y_f/Y_{fo})^{\alpha}], \quad Y_{fo} = X_{fo}/E_f \tag{177}$$

Fraction of remaining fibers at $\epsilon = R_f(\epsilon)$.

Gross stress of bundle

$$\sigma = E_f \epsilon R_f(\epsilon) \tag{178}$$

Failure occurs when $\left.\frac{d\sigma}{d\epsilon}\right|_{\epsilon = Y_B} = 0.$

$$Y_B = Y_{fo}(L\alpha)^{-1/\alpha} \tag{179}$$

Bundle strength

$$X_B = E_f Y_B R_f(Y_B) = X_{fo}(L\alpha\, e)^{-1/\alpha} = \bar{X}_f(\alpha e)^{-1/\alpha}/\Gamma(1 + 1/\alpha) \tag{180}$$

Bundle secant modulus

$$E_B = \frac{\sigma}{\epsilon} = E_f R_f(\epsilon) = E_f \exp[-L(\epsilon/Y_{fo})^{\alpha}] \tag{181}$$

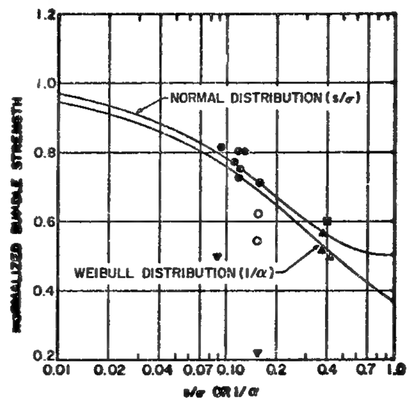


Figure 59 Normalized bundle strength vs. the inverse of shape parameter. [16]

c. Composite Strength [17]

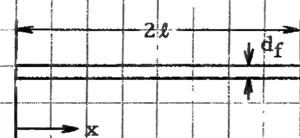
$$X = v_f \text{ (strength of bundle of length } \delta)$$

$$= v_f X_B (L/\delta)^{-1/\alpha} \tag{182}$$

$$= v_f X_{fo} (\delta \alpha e)^{-1/\alpha} \tag{183}$$

d. Determination of δ

Elastic shear lag analysis [17]



$$\frac{\sigma_f}{\sigma} = \frac{E_f}{E}\left[1 - \frac{\cosh \eta(\bar{\ell}-\bar{x})}{\cosh \eta\bar{\ell}}\right] \tag{184}$$

$$\frac{\tau}{\sigma} = \frac{\eta}{4}\,\frac{E_f}{E}\,\frac{\sinh \eta(\bar{\ell}-\bar{x})}{\cosh \eta\bar{\ell}} \tag{185}$$

$$\eta = \frac{2\sqrt{2}}{(1/\sqrt{v_f}-1)^{1/2}}\left(\frac{G_m}{E_f}\right)^{1/2}, \quad \bar{x} = x/d_f, \quad \bar{\ell} = \ell/d_f \tag{186}$$

$$\int_0^{\bar{\ell}} \sigma_f\, d\bar{x} = \sigma_f\Big|_{\bar{x}=\bar{\ell}} (\bar{\ell}-\bar{\delta}/2) \longrightarrow \bar{\delta}/2 = 1/\eta \text{ as } \bar{\ell}\to\infty. \tag{187}$$

$$\therefore \bar{\delta} = \frac{\delta}{d_f} = \left[\frac{1}{2}\,\frac{E_f}{G_m}\,\frac{1-\sqrt{v_f}}{\sqrt{v_f}}\right]^{1/2} \tag{188}$$

Perfectly plastic shear lag analysis

$$\bar{X}_f \pi d_f^2/4 = \tau_y \pi d_f \delta/2 \qquad \tau_y: \text{ matrix yield stress} \tag{189}$$

$$\therefore \bar{\delta} = \bar{X}_f/(2\tau_y) \tag{190}$$

Composite	α	d_f mm	X_{fo} MPa	E_f GPa	G_m GPa	v_f	δ(elastic) mm	X, MPa Theory	Exp	R.M.*
E-glass	6.20	0.127	3323	79.3						
Epon 815					0.024	0.095	7.73	144	71.7	150
Epon 815						0.442	3.67	755	145	681
S-glass	7.68	0.127	4413	86.2						
Epon 815					0.024	0.095	8.06	215	265	230
Epon 828					0.187	0.565	1.11	1656	1207	1365
Boron	11.11	0.102	3994	372						
Epon 815					0.024	0.061	15.68	140	132	161

*Rule of Mixtures: $X = \bar{X}_f[v_f + (1-v_f)E_m/E_f]$. Data from [18].

e. Remarks

(1) Stress concentration due to fiber breakage

(2) Dispersion of failure sites

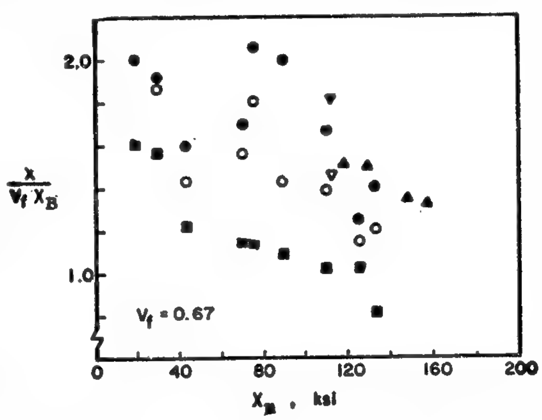


Figure 60 Normalized composite strength vs. bundle strength. [16]

SECTION V

IN-PLANE PROPERTIES OF SYMMETRIC LAMINATES

1. STRESS-STRAIN RELATIONS

a. Definition and Assumptions

(1) Symmetric Laminates

$$\begin{aligned} \theta(z) &= \theta(-z) \\ Q_{ij}(z) &= Q_{ij}(-z) \end{aligned} \tag{191}$$

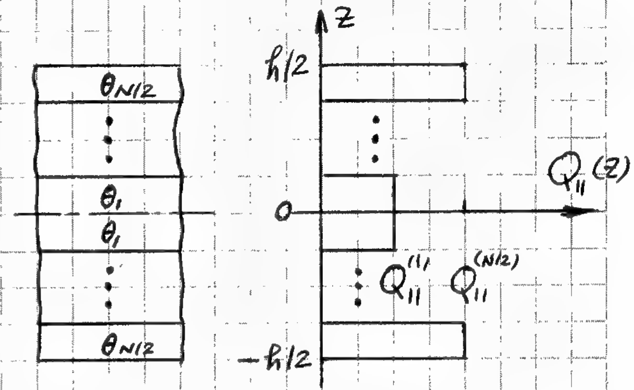


Figure 61. Ply orientations and modulus of symmetric laminates.

(2) Constant In-Plane Strain

$$\begin{aligned} e_1^o &= e_1^{(1)} = \cdots e_1^{(N/2)} \\ e_2^o &= e_2^{(1)} = \cdots e_2^{(N/2)} \\ e_6^o &= e_6^{(1)} = \cdots e_6^{(N/2)} \end{aligned} \tag{192}$$

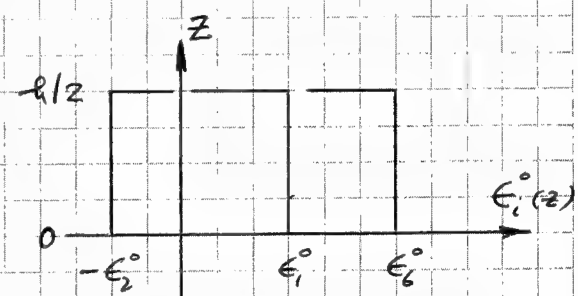


Figure 62. Assumed uniform in-plane strain across entire laminate.

(3) Average Stress

$$\bar{\sigma}_i = \frac{2}{h} \int_o^{h/2} \sigma_i \, dZ \tag{193}$$

$$= \frac{2}{h} \sum_{t=1}^{N/2} \sigma_i^{(t)} (h_t - h_{t-1}) \tag{194}$$

$$= \frac{2}{N} \sum_{t=1}^{N/2} \sigma_i^{(t)} t \tag{195}$$

where N = total number of plies

$\frac{h}{N}$ = ply thickness = h_o

or $N = \frac{h}{h_o}$

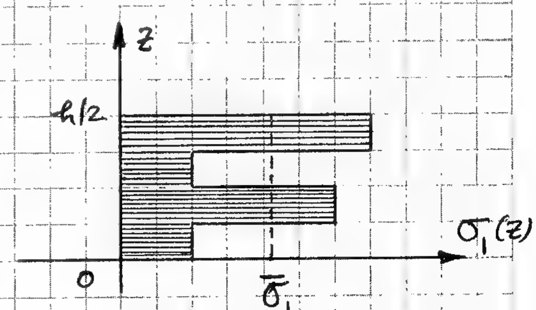


Figure 63. Average stress across laminate thickness.

(4) Stress Resultant

$$\left.\begin{aligned} N_1 &= h\,\bar{\sigma}_1 \\ N_2 &= h\,\bar{\sigma}_2 \\ N_6 &= h\,\bar{\sigma}_6 \end{aligned}\right\} \quad \text{or} \quad N_i = h\,\bar{\sigma}_i \qquad (196)$$

where N_i = Distributed load per unit width of a plate with thickness h; in Nm^{-1} or Pam

(5) In-Plane Modulus A_{ij} (same as N_i, or Nm^{-1} or Pam)

Substitute

$$\sigma_i^{(t)} = Q_{ij}^{(t)}\, e_j^{(t)} = Q_{ij}^{(t)}\, e_j^{o} \qquad (197)$$

$$N_i = 2h_o \sum_{t=1}^{N/2} Q_{ij}^{(t)}\, e_j^{o}\, t \qquad (198)$$

$$= 2h_o\, e_j^{o} \sum_{t=1}^{N/2} Q_{ij}^{(t)}\, t \qquad (199)$$

$$= A_{ij}\, e_j^{o} \qquad (200)$$

$$\text{where } A_{ij} = 2h_o \sum_{t=1}^{N/2} Q_{ij}^{(t)}\, t \qquad (201)$$

Since most practical laminates have up to 4 ply orientations,

$$A_{ij} = h_o \sum_{\alpha_1, \alpha_2, \ldots} Q^{(\alpha)}\, \eta_\alpha \qquad (202)$$

η_α = total number of plies with α orientation.

$Q_{ij}^{(\alpha)}$ = Modulus of α orientation.

A_{ij} is governed by simple rule of mixtures; stacking sequence is of no consequence. Stacking sequence is critical for other laminate properties, such as flexural rigidity.

b. In-Plane Modulus and Compliance

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} \begin{Bmatrix} e_1^o \\ e_2^o \\ e_6^o \end{Bmatrix} \quad \text{or} \quad N_i = A_{ij}\, e_j^o \tag{203}$$

$$\text{Let Compliance} = a_{ij} = [A_{ij}]^{-1} \quad \text{or} \quad a_{ij} A_{jk} = \delta_{ik} \tag{204}$$

$$\begin{Bmatrix} e_1^o \\ e_2^o \\ e_6^o \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{21} & a_{22} & a_{26} \\ a_{61} & a_{62} & a_{66} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} \quad \text{or} \quad e_i^o = a_{ij} N_j \tag{205}$$

where

$$\begin{aligned} a_{11} &= \frac{1}{\Delta}(A_{22}A_{66} - A_{26}^2), & a_{22} &= \frac{1}{\Delta}(A_{11}A_{66} - A_{16}^2) \\ a_{12} &= \frac{1}{\Delta}(A_{16}A_{26} - A_{12}A_{66}), & a_{66} &= \frac{1}{\Delta}(A_{11}A_{22} - A_{12}^2) \\ a_{16} &= \frac{1}{\Delta}(A_{12}A_{26} - A_{22}A_{16}), & a_{26} &= \frac{1}{\Delta}(A_{12}A_{16} - A_{11}A_{26}) \end{aligned} \tag{206}$$

$$\Delta = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix}$$

In-Plane Engineering Constants:

$$E_{11}^o = \frac{1}{ha_{11}}, \quad E_{22}^o = \frac{1}{ha_{22}}, \quad N_{12}^o = -\frac{a_{12}}{a_{11}}, \quad \nu_{12}^o = -\frac{a_{12}}{a_{22}}, \quad G_{12}^o = \frac{1}{ha_{66}} \tag{207}$$

Units: e_i (mm m^{-1} or mm/m)

N_i (Nm^{-1} or Pam) $\qquad a_{ij}$ $\left[N^{-1}m \text{ or} (Pam)^{-1}\right]$

A_{ij} (Nm^{-1} or Pam) $\qquad E_{11}^o$ (Nm^{-2} or Pa)

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2. FORMULAS FOR IN-PLANE MODULUS

a. General Multi-directional Laminates

$$A_{ij}/h = \frac{1}{N} \sum_{\alpha_1, \alpha_2, \ldots} Q_{ij}^{(\alpha)} \eta_\alpha \tag{208}$$

$$A_{11}/h = I_1 + I_2 + V_1 R_1 + V_2 R_2 \tag{209}$$

$$A_{22}/h = I_1 + I_2 - V_1 R_1 + V_2 R_2 \tag{210}$$

$$A_{12}/h = I_1 - I_2 - V_2 R_2 \tag{211}$$

$$A_{66}/h = I_2 - V_2 R_2 \tag{212}$$

$$A_{16}/h = -\frac{1}{2} V_3 R_1 - V_4 R_2 \tag{213}$$

$$A_{26}/h = -\frac{1}{2} V_3 R_1 + V_4 R_2 \tag{214}$$

where $$V_1 = \frac{1}{N} \sum \eta_\alpha \cos 2\alpha \tag{215}$$

$$V_2 = \frac{1}{N} \sum \eta_\alpha \cos 4\alpha \tag{216}$$

$$V_3 = \frac{1}{N} \sum \eta_\alpha \sin 2\alpha \tag{217}$$

$$V_4 = \frac{1}{N} \sum \eta_\alpha \sin 4\alpha \tag{218}$$

Since A_{ij} transforms the same way as Q_{ij}, some invariants must exist:

$$I_{1A} = \frac{1}{4}(A_{11} + A_{22} + 2A_{12}) = hI_1 \tag{219}$$

$$I_{2A} = \frac{1}{8}(A_{11} + A_{22} - 2A_{12} - 4A_{66}) = hI_2 \tag{220}$$

$$\begin{aligned} R_{1A} &= \frac{1}{2}\sqrt{(-A_{11} + A_{22})^2 + 4(A_{16} + A_{26})^2} \\ &= h\sqrt{V_1^2 + V_3^2}\, R_1 \end{aligned} \tag{221}$$

$$\begin{aligned} R_{2A} &= \frac{1}{2}\sqrt{(A_{11} + A_{22} - 2A_{12} - 4A_{66})^2 + 16(A_{16} - A_{26})^2} \\ &= h\sqrt{V_2^2 + V_4^2}\, R_2 \end{aligned} \tag{222}$$

b. In-Plane Modulus for $[0_p/90_q/45_r/-45_s]$

Simplification of ply composite's factors because

α	$\cos 2\alpha$	$\cos 4\alpha$	$\sin 2\alpha$	$\sin 4\alpha$
0	1	1	0	0
90	-1	1	0	0
45	0	-1	1	0
-45	0	-1	-1	0

$$V_1 = \frac{1}{N} (\eta_o - \eta_{90}) \tag{223}$$

$$V_2 = \frac{1}{N} (\eta_o + \eta_{90} - \eta_{45} - \eta_{-45}) \tag{224}$$

$$V_3 = \frac{1}{N} (\eta_{45} - \eta_{-45}) \tag{225}$$

$$V_4 = 0 \tag{226}$$

$$\tan 2\delta_1 = \frac{2(A_{16} + A_{26})}{A_{11} - A_{22}} = \frac{\eta_{45} - \eta_{-45}}{\eta_o - \eta_{90}} = -\frac{V_3}{V_1} \tag{227}$$

$$\tan 4\delta_2 = \frac{4(A_{16} - A_{26})}{A_{11} + A_{22} - 2A_{12} - 4A_{66}} = -\frac{V_4}{V_2} = 0 \quad \text{or} \quad \delta_2 = 0, \pm\pi, \pm 2\pi \tag{228}$$

TABLE 29 FORMULAS FOR IN-PLANE MODULUS FOR $[0_p/90_q/45_r/-45_s]$

	I_1	I_2	$\sqrt{V_1^2+V_3^2}\,R_1$	$\sqrt{V_2^2+V_4^2}\,R_2$
A'_{11}/h	1	1	$\cos 2(\theta-\delta_1)$	$\cos 4\theta$
A'_{22}/h	1	1	$-\cos 2(\theta-\delta_1)$	$\cos 4\theta$
A'_{12}/h	1	-1	0	$-\cos 4\theta$
A'_{66}/h	0	1	0	$-\cos 4\theta$
A'_{16}/h	0	0	$-\frac{1}{2}\sin 2(\theta-\delta_1)$	$-\sin 4\theta$
A'_{26}/h	0 0	0	$-\frac{1}{2}\sin 2(\theta-\delta_1)$	$\sin 4\theta$

$$\sqrt{V_1^2+V_3^2} = \frac{1}{N}\sqrt{(\eta_o-\eta_{90})^2+(\eta_{45}-\eta_{-45})^2} \tag{229}$$

$$\sqrt{V_2^2+V_4^2} = \frac{1}{N}(\eta_o+\eta_{90}-\eta_{45}-\eta_{-45}) \tag{230}$$

$$\tan 2\delta_1 = \frac{\eta_{45}-\eta_{-45}}{\eta_o-\eta_{90}} \tag{231}$$

Laminate is orthotropic if $\eta_{45} = \eta_{-45}$.

Note: $$0 \le V_1 \le 1, \qquad 0 \le V_2 \le 1 \tag{232}$$

- o At lower bound (equal to zero), the laminate is isotropic.
- o At upper bound (equal to unity), the laminate is homogeneous.
- o In general, laminates are always less anisotropic than the constituent ply.

c. Numerical Example of In-Plane Properties of T-300/5208 Laminates. For this material, the ply properties in GPa are: $I_1 = 49.5$, $I_2 = 26.9$, $R_1 = 85.7$, and $R_2 = 19.7$

TABLE 30 IN-PLANE PROPERTIES OF T-300/5208 LAMINATES

Laminates	[0/90]	[45/-45]	[0/90/45/-45]			
N	2	2	4			
$\sqrt{V_1^2 + V_3^2}$	0	0	0			
$\sqrt{V_1^2 + V_4^2}$	1	-1	0			
δ_1	0	0	0			
A_{11}/h (MPa)						
A_{22}/h "						
A_{12}/h "						
A_{66}/h "						
$A_{16}/h = A_{26}/h$						
$a_{11}h$ (MPa)						
$a_{22}h$ "						
$a_{12}h$ "						
$a_{66}h$ "						
$a_{16}h = a_{26}h$ "						
E_{11}^o (MPa)						
E_{22}^o "						
ν_{12}^o						
G_{12}^o (MPa)						

Laminates	[0/90]	[45/-45]	[0/90/45/-45]			
β^o (for $A_{16}=0$)						
δ^o (")						
$n^o = \beta^o + \delta^o$						
$k^o = \beta^o \delta^o$						

d. Difference Between Beams and Plates Theory.

3. CALCULATION OF PLY STRESSES

a. Calculation of In-Plane Strains

$$
\begin{aligned}
e_1^o &= a_{11}N_1 + a_{12}N_2 + a_{16}N_6 \\
e_2^o &= a_{21}N_1 + a_{22}N_2 + a_{26}N_6 \\
e_6^o &= a_{61}N_1 + a_{62}N_2 + a_{66}N_6
\end{aligned}
\tag{233}
$$

or

$$
\begin{Bmatrix} e_1^o \\ e_2^o \\ e_6^o \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{21} & a_{22} & a_{26} \\ a_{61} & a_{62} & a_{66} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} \quad \text{or} \quad e_i^o = a_{ij}N_j \tag{234}
$$

b. Calculation of Stresses of the t-th ply

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}^{(t)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix}^{(t)} \begin{Bmatrix} e_1^o \\ e_2^o \\ e_6^o \end{Bmatrix} \tag{235}
$$

For the α oriented ply, replace index "t" by "α" in the above equation

c.

TABLE 31 SAMPLE CALCULATIONS OF PLY STRESSES FOR T-300/5208

Laminate	Imposed N_i	Resulting e_i^o	$\sigma_i^{(o)}$	$\sigma_i^{(90)}$	$\sigma_i^{(45)}$	$\sigma_i^{(-45)}$
[0/90]	1 0 0					
[0/90]	1 1 1					
[45/-45]	1 0 0					
[45/-45]	0 0 1					
[0/90/45/-45]	1 0 0					

d. Other Examples

Laminate	N_i	e_i^o	$\sigma_i^{(o)}$	$\sigma_i^{(90)}$	$\sigma_i^{(45)}$	$\sigma_i^{(-45)}$

4. PLANE ELASTICITY SOLUTIONS FOR SYMMETRIC LAMINATES

a. Complex Parameters

Formulation and solution for laminates are identical to those for microscopically homogeneous material such as unidirectional composites. Only the complex parameters need to be exchanged:

$$\beta^o \text{ for } \beta, \quad \delta^o \text{ for } \delta, \quad n^o = n, \quad k^o \text{ for } k.$$

b. Stress Concentrations

These are applied to the average stresses in a laminate; i.e.,

$$\bar{\sigma}_i \text{ or } N_i/h$$

The actual stress concentration within each ply can be calculated by the same ply stress methodology, i.e.

$$SCF = \frac{\bar{\sigma}_\theta}{p} = \frac{N_\theta}{hp}, \quad \text{or} \quad \frac{\bar{\sigma}_r}{p} = \frac{N_r}{hp} \tag{236}$$

The in-plane strains can be calculated from, as before,

$$e_i^o = a_{ij} N_j$$

Then the ply stress are

$$\sigma_i^{(t)} = Q_{ij}^{(t)} e_j^o \quad \text{or} \quad \sigma_i^{(\alpha)} = Q_{ij}^{(\alpha)} e_j^o$$

c. TABLE 32 SAMPLE CALCULATION OF ELLIPTIC HOLE UNDER TENSION IN A T-300/5208 ORTHOTROPIC LAMINATE

$$\frac{\sigma_\theta}{p} = 1 + \frac{nb}{a}, \quad n = \beta + \delta$$

Laminate	e_i^o	$\sigma_i^{(o)}$	$\sigma_i^{(90)}$	$\sigma_i^{(45)}$	$\sigma_i^{(-45)}$
[0/90] $n =$					
[45/-45] $n =$					
[0/90/45/-45] $n =$					

d. Other Examples

SECTION VI

FLEXURAL PROPERTIES OF SYMMETRIC LAMINATES

1. FLEXURE - CURVATURE RELATIONS

a. Definitions and Assumptions

(1) Symmetric laminates, as before is assumed.

$$\theta(z) = \theta(-z)$$
$$Q_{ij}(z) = Q_{ij}(-z) \qquad (237)$$

(2) Instead varying strain across laminate thickness is assumed

$$e_1 = k_1, \quad e_2 = zk_2, \quad e_6 = zk_6 \qquad (238)$$

where k_1, k_2, k_6 are curvatures (m^{-1})

(3) Moments

$$M_i = \int_{-h/2}^{h/2} \sigma_i Z dZ \qquad (239)$$

$$= 2\int_0^{h/2} Q_{ij} e_j Z dZ \qquad (240)$$

$$= 2k_j \int_0^{h/2} Q_{ij} Z \qquad (241)$$

$$= \frac{2}{3} k_j \sum_{t=1}^{N/2} Q_{ij}^{(t)} (h_t^3 - h_{t-1}^3) \qquad (242)$$

$$M_i = D_{ij} k_j \qquad (243)$$

M_1, M_2, M_6 = distributed moments or moments per unit length (N)

D_{ij} = flexural rigidity (Nm)

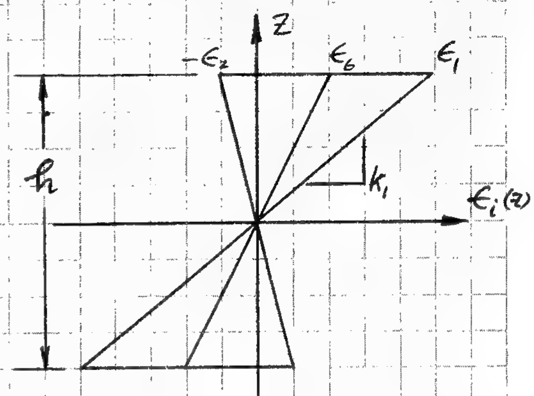


Figure 64 Linear strain variation. The slope is curvature.

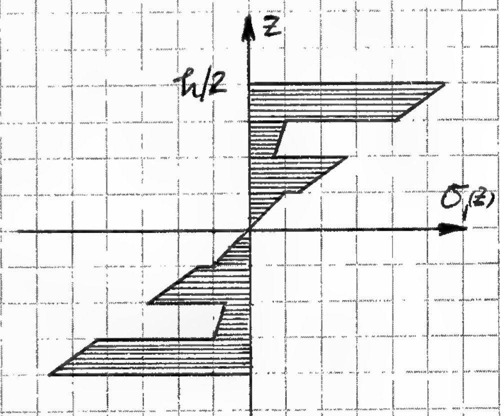


Figure 65 Ply stress is piece-wise-linear and discontinuous.

b. Flexural Rigidity and Compliance

In-plane modulus or rigidity

$$A_{ij} = \sum Q_{ij}^{(t)} (h_t - h_{t-1}) \tag{244}$$

Flexural rigidity

$$D_{ij} = \frac{1}{3} \sum Q_{ij}^{(t)} (h_t^3 - h_{t-1}^3) \tag{245}$$

A_{ij} follows a linear rule of mixtures, and independent of stacking sequence.
D_{ij} follows a weighted or cubic mixtures relation, and is highly dependent on stacking sequence.

Inverse of rigidity D_{ij} is compliance d_{ij} such that

$$\begin{Bmatrix} k_1 \\ k_2 \\ k_6 \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{21} & d_{22} & d_{26} \\ d_{61} & d_{62} & d_{66} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} \quad \text{or} \quad k_i = d_{ij} M_j \tag{246}$$

Unit of d_{ij} is $(Nm)^{-1}$

c. Engineering Constants

Similar to those for in-plane properties, engineering constants for flexural properties must also be derived from the compliance d_{ij}, not the rigidity D_{ij} because the latter can be related to engineering constants only when D_{ij} is orthotropic

Therefore $$E_{11}^f = \frac{1}{Id_{11}} = \frac{12}{h^3 d_{11}}, \quad (EI)_1 = \frac{1}{d_{11}}$$

$$E_{22}^f = \frac{12}{h^3 d_{22}} \qquad (EI)_2 = \frac{1}{d_{22}} \tag{247}$$

$$\nu_{12}^f = -\frac{d_{12}}{d_{11}}$$

$$G_{12}^f = \frac{12}{h^3 d_{66}} \tag{248}$$

2. FORMULA FOR FLEXURAL RIGIDITY

a. Derivation

For a general symmetric structure consisting of

(1) A substructure of modulus Q^o_{ij} and half depth z_o, or equivalent plies N_o (= z_o/h_o).

(2) Two symmetric facing sheets (with respect to the mid-plane of the total structure) with composite laminates of $(N/2) - N_o$ plies each.

Rearrange of formula for rigidity

$$\frac{12}{h^3} D_{ij} = \left(\frac{2 z_o}{h}\right)^3 Q^o_{ij} + \left(\frac{2}{N}\right)^3 \sum_{t=N_o+1}^{N/2} Q^t_{ij} F_t \tag{249}$$

$$\text{where } F_t = t^3 - (t-1)^3 = 3t^2 - 3t + 1 \tag{250}$$

Express in terms of invariant-form transformation

$$\frac{12}{h^3} D_{11} - \left(\frac{2 z_o}{h}\right)^3 Q^o_{11} = (I_1 + I_2) + V_1 R_1 + V_2 R_2 \tag{251}$$

$$\frac{12}{h^3} D_{22} - \left(\frac{2 z_o}{h}\right)^3 Q^o_{22} = (I_1 + I_2) - V_1 R_1 + V_2 R_2 \tag{252}$$

$$\frac{12}{h^3} D_{12} - \left(\frac{2 z_o}{h}\right)^3 Q^o_{12} = (I_1 - I_2) - V_2 R_2 \tag{253}$$

$$\frac{12}{h^3} D_{66} - \left(\frac{2 z_o}{h}\right)^3 Q^o_{66} = I_2 - V_2 R_2 \tag{254}$$

$$\frac{12}{h^3} D_{16} - \left(\frac{2 z_o}{h}\right)^3 Q^o_{16} = -\frac{1}{2} V_3 R_1 - V_4 R_2 \tag{255}$$

$$\frac{12}{h^3} D_{26} - \left(\frac{2 z_o}{h}\right)^3 Q^o_{26} = -\frac{1}{2} V_3 R_1 + V_4 R_2 \tag{256}$$

$$\text{where } V_1 = \left(\frac{2}{N}\right)^3 \sum F_t \cos 2\alpha_t, \quad V_2 = \left(\frac{2}{N}\right)^3 \sum F_t \cos 4\alpha_t \tag{257}$$

$$V_3 = \left(\frac{2}{N}\right)^3 \sum F_t \sin 2\alpha_t, \quad V_4 = \left(\frac{2}{N}\right)^3 \sum F_t \sin 4\alpha_t \tag{258}$$

TABLE 33 FORMULAS FOR FLEXURAL RIGIDITY WITH ISOTROPIC SUBSTRUCTURE

	$\left[1-\left(\frac{2Z_o}{h}\right)^3\right] I_1$	$\left[1-\left(\frac{2Z_o}{h}\right)^3\right] I_2$	$\sqrt{V_1^2+V_3^2}\ R$	$\sqrt{V_2^2+V_4^2}\ R_2$
$\frac{12}{h^3} D'_{11} - \left(\frac{2z_o}{h}\right)^3 Q^o_{11}$	1	1	$\cos 2(\theta-\delta_1)$	$\cos 4(\theta-\delta_2)$
$\frac{12}{h^3} D'_{22} - \left(\frac{2z_o}{h}\right)^3 Q^o_{11}$	1	1	$-\cos 2(\theta-\delta_1)$	$\cos 4(\theta-\delta_2)$
$\frac{12}{h^3} D'_{12} - \left(\frac{2z_o}{h}\right)^3 Q^o_{12}$	1	-1	0	$-\cos 4(\theta-\delta_2)$
$\frac{12}{h^3} D'_{66} - \frac{1}{2}\left(\frac{2z_o}{h}\right)^3 (Q^o_{11}-Q^o_{12})$	0	1	0	$-\cos 4(\theta-\delta_2)$
$\frac{12}{h^3} D'_{16}$	0	0	$-\frac{1}{2}\sin 2(\theta-\delta_1)$	$-\sin 4(\theta-\delta_2)$
$\frac{12}{h^3} D'_{26}$	0	0	$-\frac{1}{2}\sin 2(\theta-\delta_1)$	$\sin 4(\theta-\delta_2)$

$$\sqrt{V_1^2 + V_3^2} = \left(\frac{2}{N}\right)^3 \sqrt{\left(\sum F_t \cos 2\alpha_t\right)^2 + \left(\sum F_t \sin 2\alpha_t\right)^2} \qquad (259)$$

$$\sqrt{V_2^2 + V_4^2} = \left(\frac{2}{N}\right)^3 \sqrt{\left(\sum F_t \cos 4\alpha_t\right)^2 + \left(\sum F_t \sin 4\alpha_t\right)^2} \qquad (260)$$

$$\tan 2\delta_1 = \frac{2(D_{16} + D_{26})}{D_{11} - D_{22}} = -\frac{V_3}{V_1} \qquad (261)$$

$$\tan 4\delta_2 = \frac{4(D_{16} - D_{26})}{D_{11} + D_{22} - 2D_{12} - 4D_{66}} = -\frac{V_4}{V_2} \qquad (262)$$

$$F_t = 3\left(t + \frac{z_o}{h_o}\right)^3 - 3\left(t + \frac{z_o}{h_o}\right) + 1 \qquad (263)$$

(1) $\theta = 0$

TABLE 34 RIGIDITY OF CROSS-PLY LAMINATES

	$I_1 + I_2$	$\sqrt{V_1^2 + V_3^2}\ R_1$	R_2
$\frac{12}{h^3} D_{11}$	1	1	1
$\frac{12}{h^3} D_{22}$	1	-1	1

$D_{11} \rightarrow D_{22}$ as $N = \infty$

D_{12}, D_{66} not affected by stacking sequence

$D_{16} = D_{26} = 0$

(2) $\theta = \pi/4$ (This is the same as the ±45 laminate)

TABLE 35 TRANSFORMED RIGIDITY OF CROSS-PLY LAMINATES

	I_1	I_2	$\sqrt{V_1^2 + V_3^2}\ R_1$	R_2
$\frac{12}{h^3} D'_{11}$	1	1	0	-1
$\frac{12}{h^3} D'_{22}$	1	1	0	-1
$\frac{12}{h^3} D'_{12}$	1	-1	0	1
$\frac{12}{h^3} D'_{66}$	0	1	0	1
$\frac{12}{h^3} D'_{16}$	0	0	$-\frac{1}{2}$	0
$\frac{12}{h^3} D'_{26}$	0	0	$-\frac{1}{2}$	0

$D'_{11} = D'_{22}$

(no stacking sequence effect)

$D'_{16} = D'_{26}$

(has stacking sequence effect)

$D'_{16} = D'_{26} = 0$ as N

TABLE 36 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

t	F_t	α_t	$\cos 2\alpha_t$	$F_t \cos 2\alpha_t$	$\sin 2\alpha_t$	$F_t \sin 2\alpha_t$	$\cos 4\alpha_t$	$F_t \cos 4\alpha_t$	$\sin 4\alpha_t$	$F_t \sin 4\alpha_t$
1	1									
2	7									
3	19									
4	37									
5	61									
6	91									
7	127									
8	169									
9	217									
10	271									
11	331									
12	397									
13	469									
14	547									
15	631									
16	721									

h =	$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$ =	$\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =	
N =	$-V_3 / V_1$ =		$-V_4 / V_2$ =	
z_o =	δ_1 =		δ_2 =	

TABLE 37 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

		α_t	$\cos 2\alpha_t$		$\sin 2\alpha_t$		$\cos 4\alpha_t$		$\sin 4\alpha_t$	

t	F_t			$F_t \cos 2\alpha_t$	$F_t \sin 2\alpha_t$	$F_t \cos 4\alpha_t$	$F_t \sin 4\alpha_t$
1	1						
2	7						
3	19						
4	37						
5	61						
6	91						
7	127						
8	169						
9	217						
10	271						
11	331						
12	397						
13	469						
14	547						
15	631						
16	721						
$h =$				$\Sigma Fc2 =$	$\Sigma Fs2 =$	$\Sigma Fc4 =$	$\Sigma Fs4 =$
$h_o =$				$\sqrt{V_1^2 + V_3^2} =$		$\sqrt{V_2^2 + V_4^2} =$	
$N =$				$- V_3 / V_1 =$		$- V_4 / V_2 =$	
$z_o =$				$\delta_1 =$		$\delta_2 =$	

TABLE 38 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

		α_t	$\cos 2\alpha_t$		$\sin 2\alpha_t$		$\cos 4\alpha_t$		$\sin 4\alpha_t$	
t	F_t			$F_t \cos 2\alpha_t$		$F_t \sin 2\alpha_t$		$F_t \cos 4\alpha_t$		$F_t \sin 4\alpha_t$
1	1									
2	7									
3	19									
4	37									
5	61									
6	91									
7	127									
8	169									
9	217									
10	271									
11	331									
12	397									
13	469									
14	547									
15	631									
16	721									

h =	$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$ =	$\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =	
N =	$- V_3 / V_1$ =		$- V_4 / V_2$ =	
z_o =	δ_1 =		δ_2 =	

TABLE 39 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

t	F_t	α_t	$\cos 2\alpha_t$	$F_t \cos 2\alpha_t$	$\sin 2\alpha_t$	$F_t \sin 2\alpha_t$	$\cos 4\alpha_t$	$F_t \cos 4\alpha_t$	$\sin 4\alpha_t$	$F_t \sin 4\alpha_t$
1	1									
2	7									
3	19									
4	37									
5	61									
6	91									
7	127									
8	169									
9	217									
10	271									
11	331									
12	397									
13	469									
14	547									
15	631									
16	721									
$h =$				$\Sigma Fc2 =$		$\Sigma Fs2 =$		$\Sigma Fc4 =$	$\Sigma Fs4 =$	
$h_o =$				$\sqrt{V_1^2 + V_3^2} =$				$\sqrt{V_2^2 + V_4^2} =$		
$N =$				$- V_3 / V_1 =$				$- V_4 / V_2 =$		
$z_o =$				$\delta_1 =$				$\delta_2 =$		

TABLE 40 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

t	F_t	α_t	$\cos 2\alpha_t$	$F_t \cos 2\alpha_t$	$\sin 2\alpha_t$	$F_t \sin 2\alpha_t$	$\cos 4\alpha_t$	$F_t \cos 4\alpha_t$	$\sin 4\alpha_t$	$F_t \sin 4\alpha_t$
1	1									
2	7									
3	19									
4	37									
5	61									
6	91									
7	127									
8	169									
9	217									
10	271									
11	331									
12	397									
13	469									
14	547									
15	631									
16	721									

h =	$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$ =	$\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =	
N =	$- V_3 / V_1$ =		$- V_4 / V_2$ =	
z_o =	δ_1 =		δ_2 =	

TABLE 41 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

α_t	$\cos 2\alpha_t$	$\sin 2\alpha_t$	$\cos 4\alpha_t$	$\sin 4\alpha_t$

t	F_t	$F_t \cos 2\alpha_t$	$F_t \sin 2\alpha_t$	$F_t \cos 4\alpha_t$	$F_t \sin 4\alpha_t$
1	1				
2	7				
3	19				
4	37				
5	61				
6	91				
7	127				
8	169				
9	217				
10	271				
11	331				
12	397				
13	469				
14	547				
15	631				
16	721				
h =		$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$ =	$\Sigma Fs4$ =
h_o =		$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =	
N =		$- V_3 / V_1$ =		$- V_4 / V_2$ =	
z_o =		δ_1 =		δ_2 =	

TABLE 42 CALCULATION OF REDUCTION FACTORS (V's) FOR FLEXURAL RIGIDITY

		α_t	$\cos 2\alpha_t$		$\sin 2\alpha_t$		$\cos 4\alpha_t$		$\sin 4\alpha_t$	
t	F_t			$F_t \cos 2\alpha_t$		$F_t \sin 2\alpha_t$		$F_t \cos 4\alpha_t$		$F_t \sin 4\alpha_t$
1	1									
2	7									
3	19									
4	37									
5	61									
6	91									
7	127									
8	169									
9	217									
10	271									
11	331									
12	397									
13	469									
14	547									
15	631									
16	721									
h =				$\Sigma Fc2$ =		$\Sigma Fs2$ =		$\Sigma Fc4$ =	$\Sigma Fs4$ =	
h_o =				$\sqrt{V_1^2 + V_3^2}$ =				$\sqrt{V_2^2 + V_4^2}$ =		
N =				$-V_3 / V_1$ =				$-V_4 / V_2$ =		
z_o =				δ_1 =				δ_2 =		

b. Stacking Sequence Effect on Symmetric Cross-ply Composites

Assume no substructure; i.e., $z_o = 0$.

TABLE 43 STACKING SEQUENCE EFFECT OF CROSS-PLY LAMINATES

Laminates	$\sqrt{V_1^2 + V_3^2}$	δ_1	$\sqrt{V_2^2 + V_4^2}$	δ_2
$[0_m/90_m]_s$	0.75	0	1	0
$[0_m/90_m]_{2s}$	0.375 $(= \frac{.75}{2})$	0	1	0
$[0_m/90_m]_{4s}$	0.1875 $(= \frac{.75}{4})$	0	1	0
$[0_2/90_4/0_2]_s$	0.1875	0	1	0
$[0/90_2/0_2/90_2/0]_s$	0.0468	0	1	0
$[0/90]_\infty$	0 $(= \frac{.75}{\infty})$	0	1	0

* m = any positive integer, and has no effect on stacking sequence, but strong effect on D_{ij} values.

TABLE 44 FLEXURAL RIGIDITY OF CROSS-PLY LAMINATES

	I_1	I_2	$\sqrt{V_1^2 + V_3^2}$ R_1	R_2
$\frac{12}{h^3} D'_{11}$	1	1	$\cos 2\theta$	$\cos 4\theta$
$\frac{12}{h^3} D'_{22}$	1	1	$-\cos 2\theta$	$\cos 4\theta$
$\frac{12}{h^3} D'_{12}$	1	-1	0	$-\cos 4\theta$
$\frac{12}{h^3} D'_{66}$	0	1	0	$-\cos 4\theta$
$\frac{12}{h^3} D'_{16}$	0	0	$-\frac{1}{2}\sin 2\theta$	$-\sin 4\theta$
$\frac{12}{h^3} D'_{26}$	0	0	$-\frac{1}{2}\sin 2\theta$	$\sin 4\theta$

c. Flexural Rigidity of T-300/5208 Cross-Ply Plates and Beams

$I_1 = 49.5$ $I_2 = 26.9$ $R_1 = 85.8$ $R_2 = 19.7$ (GPa)

$h_o = .005 \times 25.4 = .125$ mm.

TABLE 45 RIGIDITY OF T-300/5208 CROSS-PLY LAMINATES

Laminates	D_{ij} (Nm)			a_{ij} $(kNm)^{-1}$	Eng'g Constants (GPa)
$[0_4/90_4]_s$ h = 2mm	107.0	1.93	0		$E_{11}^f =$
		21.2	0		$E_{22}^f =$
			4.8		$\nu_{12}^f =$
					$G_{12}^f =$
$[0_{16}/90_{16}]_s$ h= 8mm	6848	124	0		$E_{11}^f =$
		1357	0		$E_{22}^f =$
			307		$\nu_{12}^f =$
					$G_{12}^f =$
$[0/90]_{4s}$ h = 2mm	74.8	1.93	0		
		53.3	0		
			4.8		
$[0/90_2/0_2/90_2/0]_s$ h = 2mm	66.7	1.93	0		
		61.4	0		
			4.8		
$[0/90]_\infty$ h = 2mm	64.0	1.93	0		
		64.0	0		
			4.8		
$[45_4/-45_4]_s$ h = 2mm	37.8	28.2	-21.4		$E_{11}^f = E_{22}^f =$
		37.8	-21.4		
			31.1		$\nu_{12}^f =$
					$G_{12}^f =$

d. Numerical Examples

(1) Three Point Bend Test of T-300/5208 : $[0_{16}/90_{16}]_s$, h = 8mm, L = 10cm, b = 2cm

$$D_{ij} = \begin{bmatrix} 6848 & 186 & 0 \\ & 1357 & 0 \\ & & 307 \end{bmatrix} \text{(Nm)} \qquad d_{ij} = \begin{bmatrix} .146 & -.020 & 0 \\ & .740 & 0 \\ & & 3.26 \end{bmatrix} (\text{kNm})^{-1}$$

(a) Beam along 0^o on outside

$$(EI)_1 = \frac{b}{d_{11}} = .137\ (\text{kNm}^2) \qquad (264)$$

$$\frac{P}{\delta} = \frac{48EI}{L^3} = 6576\ \text{kNm}^{-1} \qquad (265)$$

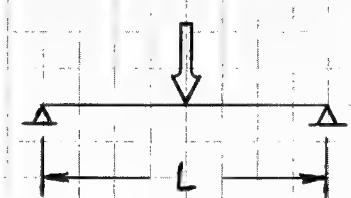


Figure 66 Three-point bend test.

(i) Load at First-Ply Failure (FPF)

$$e_T = 3.9\ \text{mm/m} \qquad (266)$$

$$z = 2\text{mm} \qquad (267)$$

$$\therefore k = \frac{e_T}{z} = 1.95\ \text{m}^{-1} \qquad (268)$$

$$M = D_{11}k_1 = 6848 \times 1.95 = 13.4\text{kN/unit width}$$

$$\therefore P_{FPF} = \frac{2Mb}{L} = \frac{2 \times 13.4 \times .02}{.05} = 10.72\ \text{kN}$$

$$\delta_{FPF} = \frac{10.72}{6576} = 1.63\ \text{mm} \qquad (269)$$

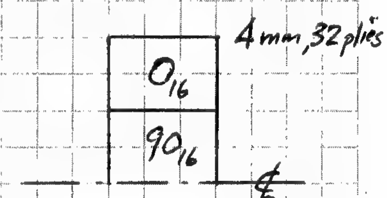


Figure 67 1-direction of [0/90].

(ii) Load at Ultimate, assuming X = 1503 MPa or e_{Ult} = 8.3 mm/m

$$k_{Ult} = \frac{e_{Ult}}{4} = 2.08\ \text{m}^{-1} \qquad (270)$$

$$M_{Ult} = 6848 \times 2.08 = 14.2\text{kN} \qquad (271)$$

$$P_{Ult} = 11.36\ \text{kN}, \quad \delta_{Ult} = 1.73\ \text{mm}. \qquad (272)$$

(b) Beam along 90° out side

$$(EI)_2 = \frac{b}{d_{22}} \tag{273}$$

$$\frac{P}{\delta} =$$

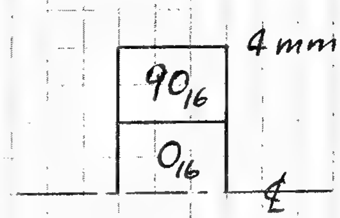


Figure 68 2-direction of [0/90].

(i) Load at First-Ply Failure (FPF)

(ii) Load at Ultimate

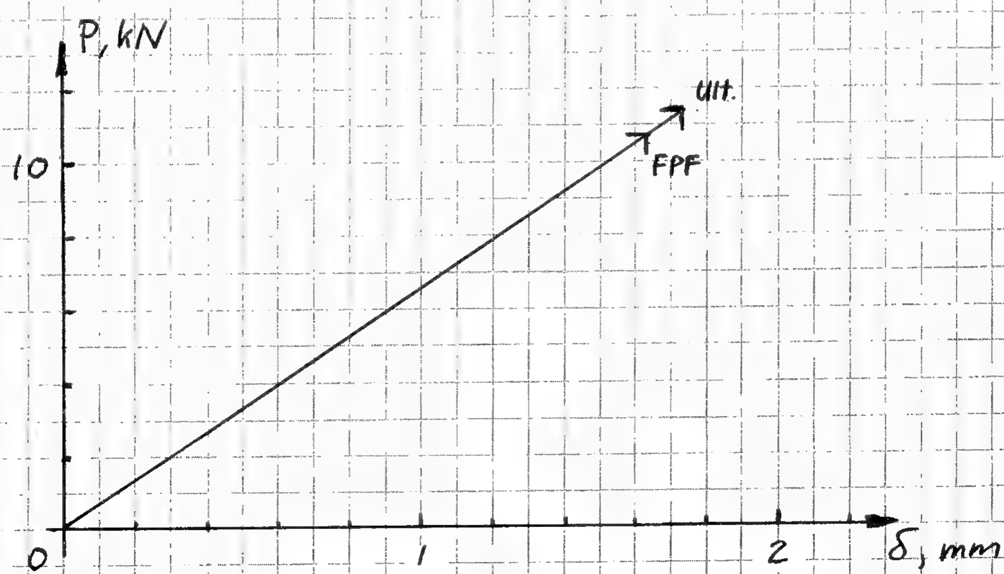


Figure 69 Load deflection curves of 0° and 90° beams of $[0_{16}/90_{16}]_s$.

e. Natural Frequencies of Transverse Vibrations of Beams

$$\omega_n = \frac{\lambda_n^2}{L^2} \frac{1}{\sqrt{d_{11}\mu}} \tag{273}$$

TABLE 46 FREE VIBRATIONS OF BEAMS WITH UNIFORM CROSS-SECTIONS

End Conditions	λ_1	λ_2	λ_3	λ for large n
Clamped-Free	1.875	4.694	7.855	(2n- 1) /2
Hinged-Hinged	3.142	6.283	9.425	$n\pi$
Clamped-Hinged	3.927	7.068	10.210	(4n+ 1) /4
Free-Hinged	"	"	"	"
Clamped-Clamped	4.730	7.853	10.996	(2n+ 1) /2
Free-Free	"	"	"	"

Example: T-300/5208 beams from $[0_{16}/90_{16}]_s$ plate

$L = 10$cm, $b = 2$cm, $h = .8$cm, density $= 1600$ kgm^{-3}

μ = mass/unit length = 0.256 kg/m

$d_{11} = .146$ kNm^{-1}, $\frac{b}{d_{11}} = 137$ Nm^2

$$\omega_n = \lambda_n^2 \frac{\sqrt{137}}{.1^2\sqrt{.256}} = 2313\lambda_n^2 \quad s^{-1} \tag{274}$$

$$\text{For hinged-hinged, } \omega_1 = {}^2 \times 2313 = 22832 \quad s^{-1} \tag{275}$$

By heating 3-point bend test as a 1-degree of freedom approximation

$$\omega_1 = \sqrt{\frac{P}{\delta M}} = \sqrt{\frac{6576000}{.025612}} = 22666 \quad s^{-1} \tag{276}$$

The difference by 2 methods is negligible as expected.

The effective mass M is one half of total mass of beam for a beam under 3-point bending.

f. Special Isotropic Homogeneous Plates Consisting of Orthotropic Plies

TABLE 47 SPECIAL ISOTROPIC HOMOGENEOUS PLATE

$$[-60/0/60_2/0/-60/60/0/-60/0/60]_s$$

For T-300/5208 material, $I_1 = 49.5$, $I_2 = 26.9$, $R_1 = 85.8$, $R_2 = 19.7$ (GPa)

$$V_1 = V_2 = \frac{1}{144}, \; V_3 = V_4 = 0, \; \delta_1 = \delta_2 = 0$$

	I_1	I_2	$\frac{1}{144} R_1$	$\frac{1}{144} R_2$
$\frac{12}{h^3} D_{11}$	1	1	$\cos 2\theta$	$\cos 4\theta$
$\frac{12}{h^3} D_{12}$	1	-1		$-\cos 4\theta$
$\frac{12}{h^3} D_{16}$	0	0	$\frac{1}{2} \sin 2\theta$	$-\sin 4\theta$

	0	15	30	45	60	75	90	Theory
$\frac{12}{h^3} D_{11}$	77.1	77.0	76.6	76.3	76.0	75.9	75.9	76.4
$\frac{12}{h^3} D_{12}$	22.5	22.5	22.7	22.7				22.6
$\frac{12}{h^3} D_{66}$	26.8	26.8	27.0	27.0				26.9
$\frac{12}{h^3} D_{16}$	0	.03	.14	.30	.38	.27	0	0

Since stacking sequence is not important for in-plane modulus A_{ij}, this special laminate is equivalent to $[0/60/-60]_{4s}$, from which

$$\sum \cos 2\alpha_t = \sum \sin 2\alpha_t = \sum \cos 4\alpha_t = \sum \sin 4\alpha_t = 0 \quad (277)$$

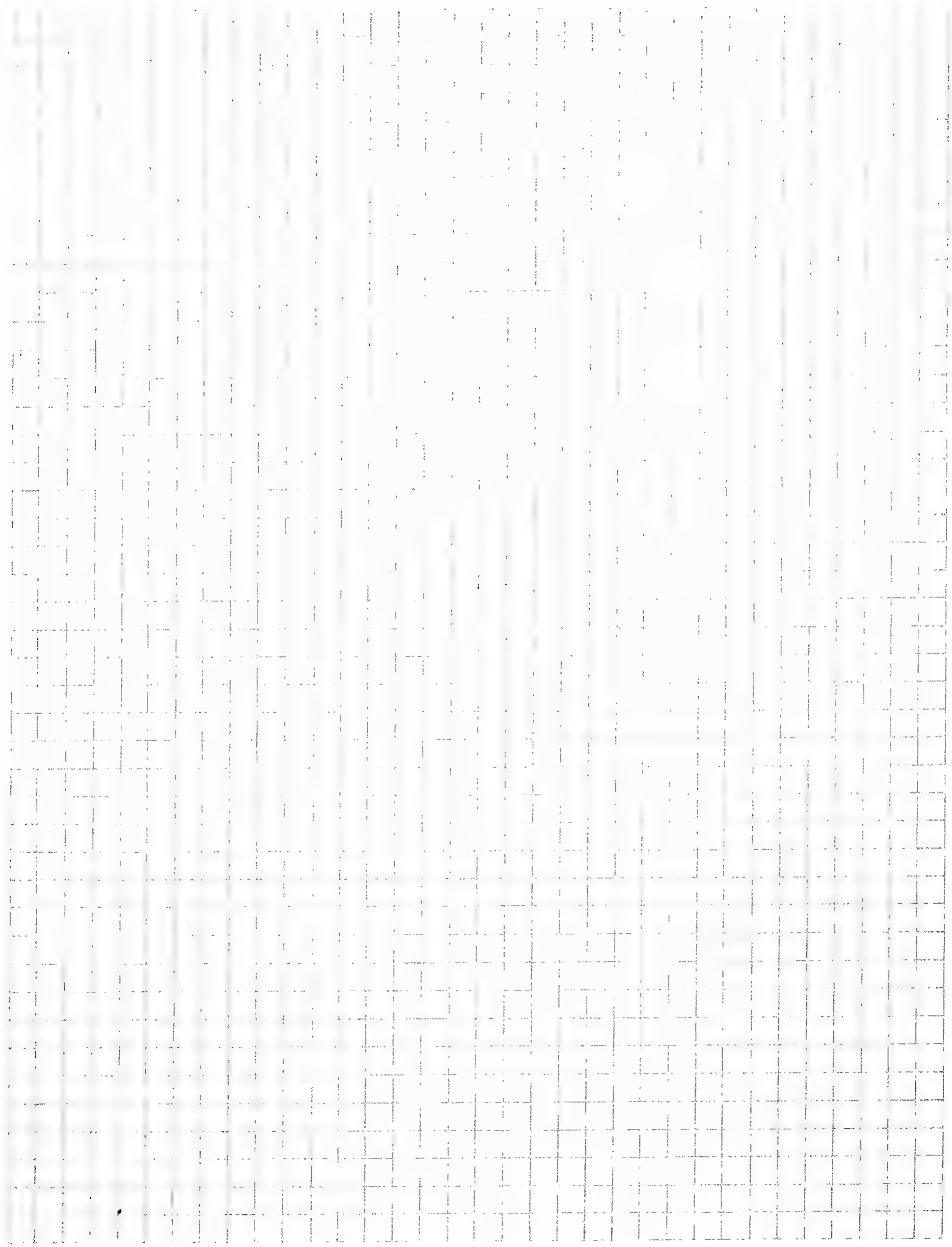
Then $A_{11}/h = A_{22}/h = I_1 + I_2$

$$A_{12}/h = I_1 - I_2, \quad A_{66}/h = I_2 \qquad (278)$$

$$A_{16} = A_{26} = 0$$

Then A_{ij} is isotropic

Since $A_{ij} = \frac{12}{h^2} D_{ij}$ for this plate, the laminate is also homogeneous.



3. FORMULA FOR RIGIDITY FOR SANDWICH PLATES

a. General Considerations

Assuming: Core stiffness $Q^{o}_{ij} = 0$

Facing material = homogeneous (not laminated)

$$D_{ij} = \frac{2}{3} Q^{F}_{ij}\left[\left(\frac{h}{2}\right)^3 - z_o^3\right] \quad , \quad \text{or} \tag{279}$$

$$\frac{12}{h^3}\frac{D_{ij}}{Q^{F}_{ij}} = 1 - \left(\frac{2z_o}{h}\right)^3 \tag{280}$$

Density of sandwich:

$$\frac{\rho}{\rho_F} = 1 - \frac{2z_o}{h}\left(1 - \frac{\rho_c}{\rho_F}\right)$$

$$\rho_c \leq 500 \text{ kg m}^{-2}$$

$$\frac{\rho_c}{\rho_F} \leq .2$$

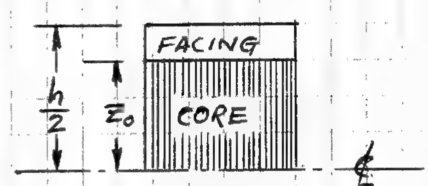


Figure 70 Sandwich Plate.

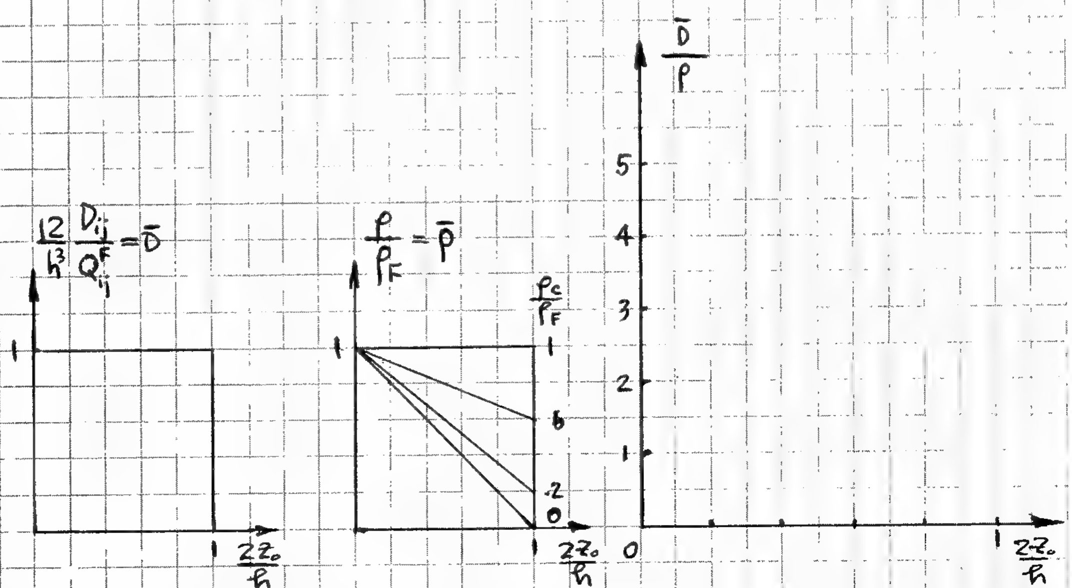


Figure 71 Specific rigidity as function of percentage of core. Note optimum combinations of density and stiffness ratios.

b. Flexural Rigidity of T-300/5208 Sandwich Plates

$I_1 = 49.5$, $I_2 = 26.9$, $R_1 = 85.8$, $R_2 = 19.7$ MPa

$z_o = 1$ mm (or $8\,h_o$), $h_o = .125$ mm, $h = 4$ mm (or $32\,h_o$)

$Q^o_{ij} = 0$. $1 - \left(\frac{2z_o}{h}\right)^3 = \frac{7}{8}$, $\frac{h^3}{12} = 5.33\ (\text{mm})^3$

TABLE 48 RIGIDITY OF $[(-45/45/90/0)_2/(z_o)_8]_s$ (NON-SYMMETRICAL FACE SHEET)

	$\frac{7}{8}\ I_1 = 43.3$	$\frac{7}{8}\ I_2 = 23.5$	$.050\ R_1 = 4.29$	$.14\ R_2 = 2.76$
$\frac{12}{h^3}\ D'_{11}$	1	1	$\cos 2(\theta - 25)$	$\cos 4\theta$
$\frac{12}{h^3}\ D'_{22}$	1	1	$-\cos 2(\theta - 25)$	$\cos 4\theta$
$\frac{12}{h^3}\ D'_{12}$	1	-1	0	$-\cos 4\theta$
$\frac{12}{h^3}\ D'_{66}$	0	1	0	$-\cos 4\theta$
$\frac{12}{h^3}\ D'_{16}$	0	0	$-\frac{1}{2}\sin 2(\theta - 25)$	$-\sin 4\theta$
$\frac{12}{h^3}\ D'_{26}$	0	0	$-\frac{1}{2}\sin 2(\theta - 25)$	$\sin 4\theta$

$$\left. D_{ij} \right]_{\theta = 0} = \begin{bmatrix} 385 & 90.8 & 8.6 \\ & 356 & 8.6 \\ & & 110 \end{bmatrix} (10^{-3}\,\text{Nm}) \qquad d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$\left. D_{ij} \right]_{\theta = 25} = \begin{bmatrix} 376 & 108 & -14.5 \\ & 331 & 14.5 \\ & & 128 \end{bmatrix} (10^{-3}\,\text{Nm}) \qquad d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

TABLE 49 RIGIDITY OF $[-45/45/90/0_2/90/45/-45/(z_o)_8]_s$ (SYMMETRICAL FACE SHEET)

	$\frac{7}{8}$ $I_1 = 43.3$	$\frac{7}{8}$ $I_2 = 23.5$	$.00926$ $R_1 = 0.794$	$.0234$ $R_2 = 0.461$
$\frac{12}{h^3} D'_{11}$	1	1	$\cos 2(\theta-36)$	$\cos 4\theta$
$\frac{12}{h^3} D'_{22}$	1	1	$-\cos 2(\theta-36)$	$\cos 4\theta$
$\frac{12}{h^3} D'_{12}$	1	-1	0	$-\cos 4\theta$
$\frac{12}{h^3} D'_{66}$	0	1	0	$-\cos 4\theta$
$\frac{12}{h^3} D'_{16}$	0	0	$-\frac{1}{2}\sin 2(\theta-36)$	$-\sin 4\theta$
$\frac{12}{h^3} D'_{26}$	0	0	$-\frac{1}{2}\sin 2(\theta-36)$	$\sin 4\theta$

$$D_{ij}\Big]_{\theta=0} = \begin{bmatrix} 360 & 103 & 2.0 \\ & 357 & 2.0 \\ & & 123 \end{bmatrix} (10^{-3}\,\text{Nm}) \qquad d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$D_{ij}\Big]_{\theta=36} = \begin{bmatrix} 358 & 108 & -1.4 \\ & 350 & 1.4 \\ & & 127 \end{bmatrix} (10^{-3}\,\text{Nm}) \qquad d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

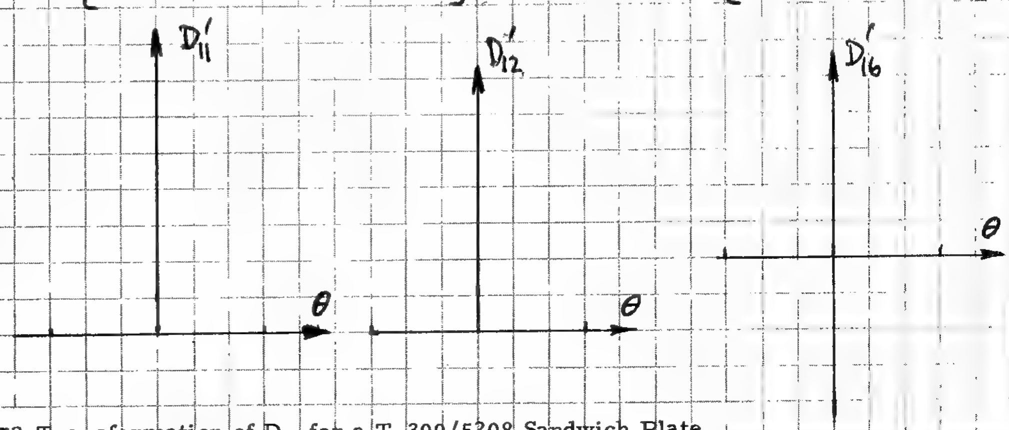


Figure 72 Transformation of D_{ij} for a T-300/5208 Sandwich Plate.

c. Sample Problems

4. DETERMINATION OF STRESSES

a. Basic Relations

TABLE 50 FLEXURE-CURVATURE RELATIONS

	k_1	k_2	k_6
M_1	D_{11}	D_{12}	D_{16}
M_2	D_{21}	D_{22}	D_{26}
M_6	D_{61}	D_{62}	D_{66}

	M_1	M_2	M_6
k_1	d_{11}	d_{12}	d_{16}
k_2	d_{21}	d_{22}	d_{26}
k_6	d_{61}	d_{62}	d_{66}

Since $e_1 = zk_1$, $e_2 = zk_2$, $e_6 = zk_6$

Finally, from stress-strain relation of the t-th layer.

$$\left.\begin{aligned} \sigma_1 &= Q_{11}e_1 + Q_{12}e_2 + Q_{16}e_6 \\ \sigma_2 &= Q_{21}e_1 + Q_{22}e_2 + Q_{26}e_6 \\ \sigma_6 &= Q_{61}e_1 + Q_{62}e_2 + Q_{66}e_6 \end{aligned}\right\} \qquad (281)$$

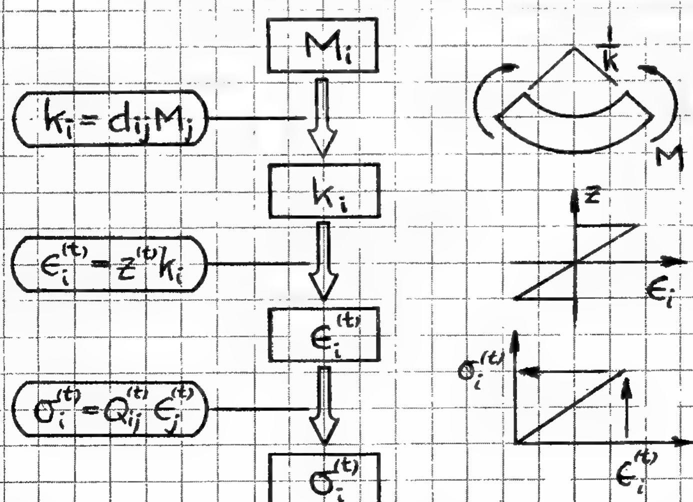


Figure 73 Flow diagram for ply stress calculation.

b. Numerical Example for Ply Stress Calculation

For T-300/5208 sandwich plate, $[-45/45/90/0_2/90/45/-45/(z_o)_8]_s$

$$D_{ij} = \begin{bmatrix} 360 & 103 & 2 \\ & 357 & 2 \\ & & 123 \end{bmatrix} (10^3 \text{Nm}) \tag{282}$$

Determination of ply stresses under $M_i' = \left\{ \quad \right\}$ (283)

(1) Inverse of D_{ij}:

$$d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \tag{284}$$

(2) Curvature $k_i = d_{ij} M_j = \left\{ \quad \right\}$ (285)

(3) Strain $e_i = zk_i$ (286)

TABLE 51 STRAIN VARIATION FROM PLY TO PLY

t	t-1	z(mm)	e_1	e_2	e_6
8	9	1.			
9	10	1.125			
10	11	1.25			
11	12	1.375			
12	13	1.5			
13	14	1.625			
14	15	1.750			
15	16	1.875			

(4) Stress $\sigma_i^{(t)} = Q_{ij}^{(t)} e_j^{(t)}$

TABLE 52 PLY STRESS AT VARIOUS LOCATIONS

t	z_t(mm)	Stresses at z_{t-1}			Stresses at z_t		
		σ_1	σ_2	σ_6	σ_1	σ_2	σ_6
9	1.125						
10	1.25						
11	1.375						
12	1.5						
13	1.625						
14	1.75						
15	1.875						
16	2.						

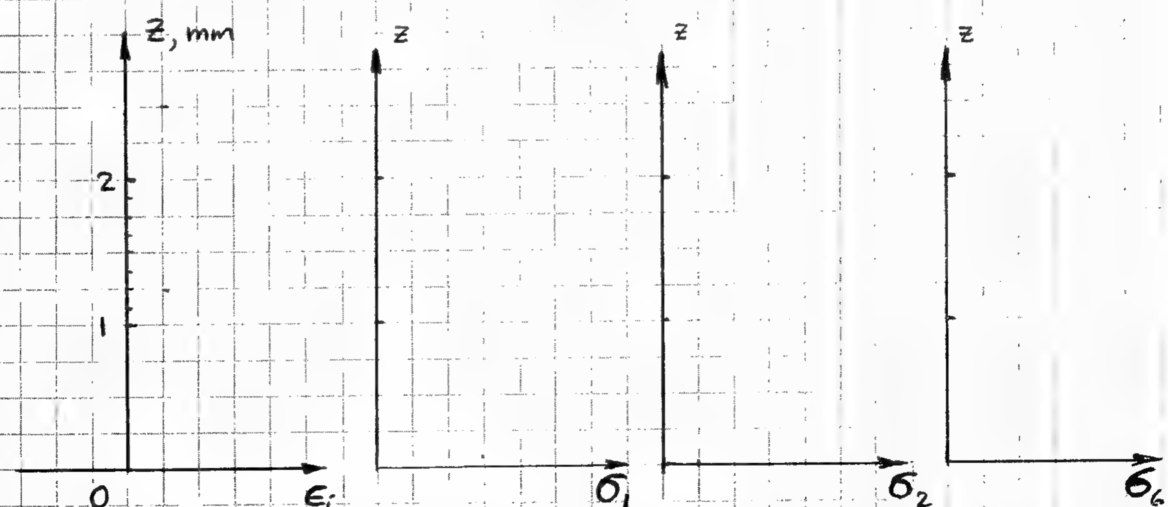


Figure 74 Stresses and Strain across laminate thickness.

c. Numerical Example for Ply Stress Calculation

$$D_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} (10^3 Nm) \qquad (287)$$

$$\text{Determination of ply stresses under } M_i = \begin{Bmatrix} \\ \\ \end{Bmatrix} \qquad (288)$$

(1) Inverse of D_{ij}:

$$d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad (289)$$

(2) Curvature $k_i = d_{ij}M_j = \begin{Bmatrix} \\ \\ \end{Bmatrix}$ (290)

(3) Strain $e_i = zk_i$

TABLE 53 STRAIN VARIATION FROM PLY TO PLY

t	t-1	z(mm)	e_1	e_2	e_6

(4) Stress $\sigma_i^{(t)} = Q_{ij}^{(t)} e_j^{(t)}$

TABLE 54 PLY STRESSES AT VARIOUS LOCATIONS

t	z_t (mm)	Stresses at z_{t-1}			Stresses at z_t		
		σ_1	σ_2	σ_6	σ_1	σ_2	σ_6

Figure 75 Stresses and Strain across laminate thickness.

d. Stress Concentration of Orthotropic Plates with a Filled Hole

Assume a cross-ply T-300/5208 laminate

$[0_{16}/90_{16}]$

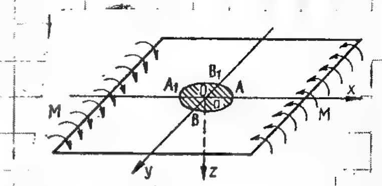


Figure 76 Bending of plate with a hole filled with rigid core.

(1) Elastic Constants

From previous calculations

$$D_{ij} = \begin{bmatrix} 6848 & 186 & 0 \\ & 1357 & 0 \\ & & 307 \end{bmatrix} \text{(Nm)}$$

$$E_{11}^f = 160, \quad E_{22}^f = 31.6, \quad \nu_{12}^f = .136, \quad G_{12}^f = 7.2 \text{ (MPa)} \tag{291}$$

Complex parameters are roots of equations

$$D_{22}\mu^4 + 2(D_{12} + 2D_{66})\mu^2 + D_{11} = 0 \tag{292}$$

$$\mu^2 = -.590 \pm 2.167_i = 2.246 \left[\cos(\pm 1.30) + i \sin(\pm 1.30)\right]$$

$$\mu = \pm 1.50 \left[\cos(\pm .65) + i \sin(\pm .65)\right]$$

$$= \pm 1.50 \left[.80 \pm .60i\right] = \pm (1.2 \pm 09i) \tag{293}$$

$$n = i(\mu_1 + \mu_2) = \pm 2.4i, \pm 1.8 \tag{294}$$

$$k = -\mu_1\mu_2 = 1.2^2 + .9^2 = 2.25 \tag{295}$$

(2) Bending Moments

(a) At point A

$$M_r = \frac{M}{1 - \nu_{12}\nu_{21}} \left(1 + \frac{\nu_{12} + n}{k}\right) = 1.86 \ M$$

$$M_\theta = \nu_{21} M_r = .050M, \quad M_{r\theta} = 0 \tag{296}$$

(b) At point B

$$M_r = -\frac{M}{1 - \nu_{12}\nu_{21}} \frac{D_2}{D_1} (\nu_{12} + \nu_{12} n + k) = -.523M$$

$$M_\theta = \nu_{12} M_r = -.0712M, \quad M_{r\theta} = 0 \tag{297}$$

(3) Ply Stresses

From previous calculations

$$d_{ij} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad (298)$$

(a) At Point A

$$M_i = \begin{Bmatrix} 1.86 \\ .50 \\ 0 \end{Bmatrix} M, \quad \text{then } k_1 = \begin{Bmatrix} \\ \\ \end{Bmatrix} M \qquad (299)$$

At z = 4mm (outer most ply)

$$e_i = 4k_i = \begin{Bmatrix} \\ \\ \end{Bmatrix} M, \qquad \sigma_i = Q_{ij}^{(0)} e_j = \begin{Bmatrix} \\ \\ \end{Bmatrix} M \qquad (300)$$

At z = 2mm (Outermost 90° ply)

$$e_i = 2k_i = \begin{Bmatrix} \\ \\ \end{Bmatrix} M, \qquad \sigma_i = Q_{ij}^{(90)} e_j = \begin{Bmatrix} \\ \\ \end{Bmatrix} M \qquad (301)$$

$$M_{FPF} = \qquad\qquad M_{Ult} = \qquad (302)$$

(b) At Point B

$$M_i = \begin{Bmatrix} -.523 \\ -.071 \\ 0 \end{Bmatrix} M, \quad \text{then } k_i = \begin{Bmatrix} \\ \\ \end{Bmatrix} M \qquad (303)$$

$$\text{At } z = 4\text{mm}, \quad e_i = 4k_i = \begin{Bmatrix} \\ \\ \end{Bmatrix} M, \quad \sigma_i = Q_{ij}^{(0)} e_j = \begin{Bmatrix} \\ \\ \end{Bmatrix} M \qquad (304)$$

$$\text{At } z = 2\text{mm}, \quad e_i = 2k_i = \begin{Bmatrix} \\ \\ \end{Bmatrix} M, \quad \sigma_i = Q_{ij}^{(90)} e_j = \begin{Bmatrix} \\ \\ \end{Bmatrix} M \qquad (305)$$

$$M_{FPF} = \qquad\qquad M_{Ult} = \qquad (306)$$

SECTION VII

PROPERTIES OF UNSYMMETRICAL LAMINATES

1. BACKGROUND

An unsymmetrical laminate is one that does not have mid-plane symmetry in the layup of plies. This type of laminate has not been used extensively in actual structures, but may be considered if special constraints or effects, such as minimum gage, aeroelastic tailoring, and bimetallic set-up are desired. Unsymmetric laminates will warp after curing and cooling. The degree of warpage will change by temperature and moisture absorption, in addition to applied loads.

The properties of unsymmetric laminates are a little more complicated than the symmetric ones. Because of the lack of symmetry, the laminate will bend or twist when an in-plane load is applied or it will stretch when a moment is applied. This coupled response is unique and has no counterpart in symmetric laminates. Unsymmetrical structures, however, are not uncommon in real life. Floor slabs, fuselage, and many other built-up structures are usually unsymmetrical. Thus, the theory of unsymmetrical laminates will be discussed. They may uniquely fulfill requirements not possible with symmetric laminates.

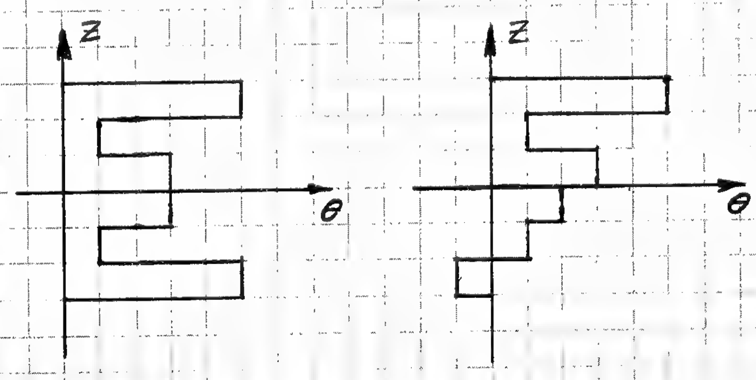
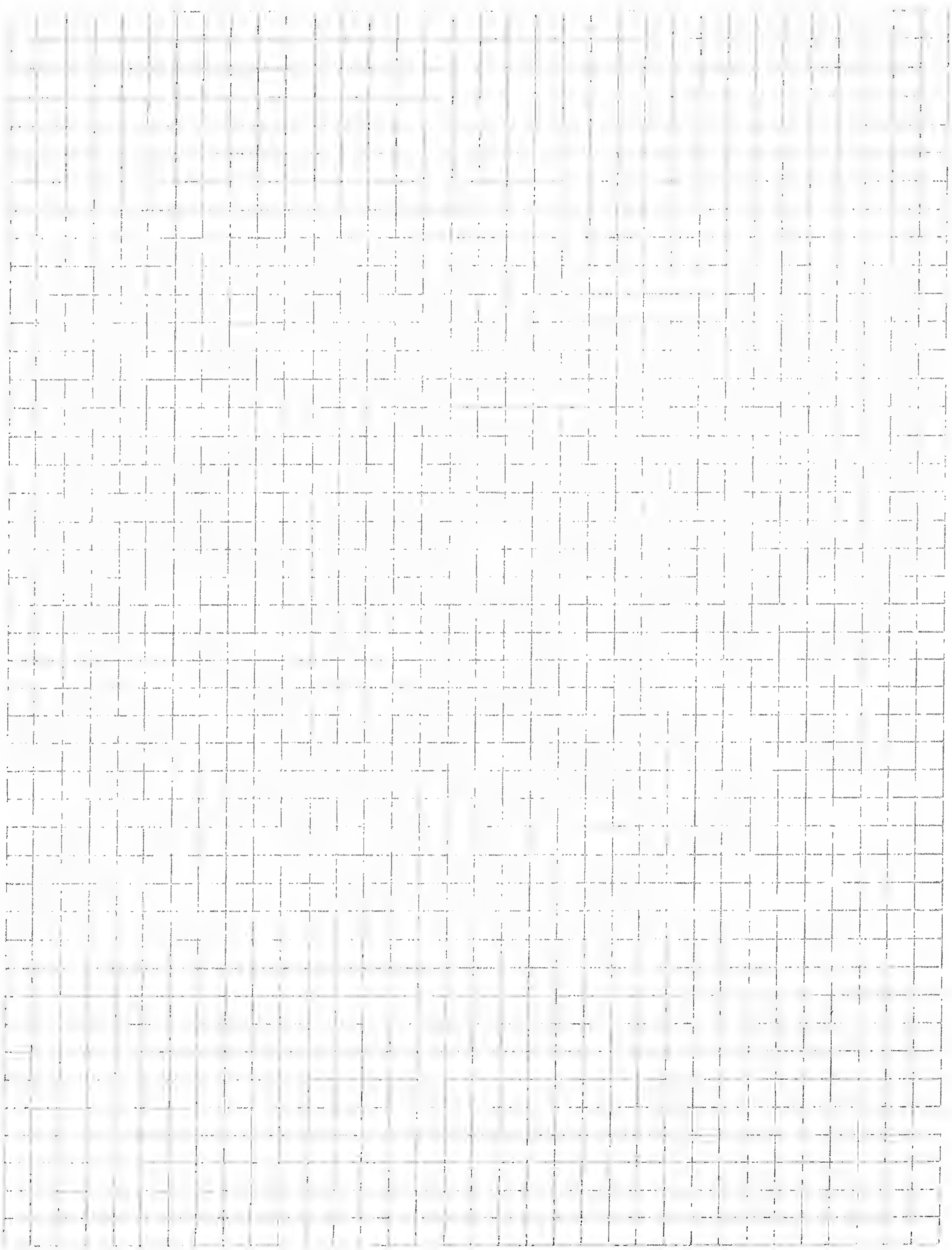


Figure 77 Symmetric versus unsymmetric laminates. Symmetry is based on mid or δ = 0 plane.



2. COUPLING MODULUS

a. Definition: In symmetric laminates, in-plane and flexural behavior are independent of each other; i.e., they are not coupled. But for unsymmetric laminates, they are coupled; i.e., in-plane extension requires both in-plane stresses as well as moments (to keep the laminate from warping), and conversely, the bending of a laminate induces both in-plane stresses and moments. The coupling modulus can be defined as follows

$$N_i = \int_{-h/2}^{h/2} \sigma_i dz \qquad (307)$$

$$= \int Q_{ij} e_j dz \qquad (308)$$

$$\text{If } e_i = e_i^o + zk_i \qquad (309)$$

$$N_i = \int Q_{ij}(e_j^o + 2k_j)dz = e_j \int Q_{ij} dz + k_j \int Q_{ij} zdz \qquad (310)$$

$$= A_{ij}N_j + B_{ij}k_j$$

Alternatively,

$$M_i = \int_{-h/2}^{h/2} \sigma_i zdz = \int Q_{ij} e_j zdz \qquad (311)$$

$$\text{If } e_i = e_i^o + zk_i$$

$$M_i = \int Q_{ij}(e_j^o + zk_j)dz = e_j^o \int Q_{ij} zdz + k_j \int Q_{ij} z^2 dz \qquad (312)$$

$$= B_{ij}e_j^o + D_{ij}M_j$$

Thus, the coupling modulus is the same between N and k as that between M and ϵ^o. Needless to say, B_{ij} is identically zero for symmetric laminates.

- For $[0/90]_T$, $-B_{11} = B_{22}$, $B_{12} = B_{66} = B_{16} = B_{26} = 0$.
- For $[\theta/-\theta]_T$, B_{16}, $B_{26} \neq 0$, $B_{11} = B_{22} = B_{12} = B_{66} = 0$.

b. Formulas for Coupling Modulus

$$B_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} z dz$$

$$= \frac{1}{2} \sum_{t=1-N/2}^{N/2} Q_{ij}^{(t)} (h_t^2 - h_{t-1}^2) \qquad (313)$$

For N plies with uniform thickness h_o, (N even)

$$B_{ij} = \frac{h_o^2}{2} \sum_{1-(N/2)}^{N/2} Q_{ij}^{(t)} \left[t^2 - (t-1)^2\right]$$

$$= \frac{h_o^2}{2} \sum Q_{ij}^{(t)} (2t-1) \qquad (314)$$

$$\frac{2}{h^2} B_{ij} = \left(\frac{1}{N}\right)^2 \sum Q_{ij}^{(t)} F_t \qquad (315)$$

$$F_t = 2t-1$$

Unlike A_{11} and D_{11}, B_{11} can be positive or negative depending on the shifting of the neutral plane up or down.

Substituting transformation of Q_{ij}, B_{ij} can be defined.

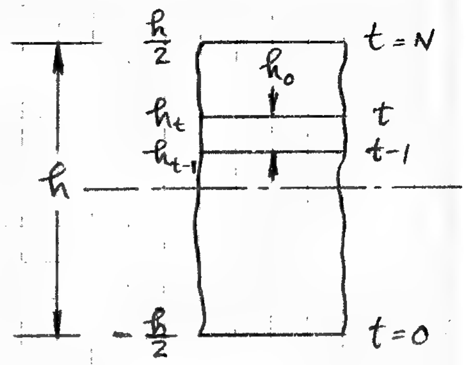


Figure 78 Integration of unsymmetric laminates must be performed for the entire thickness from -h/2 to h/2. (For symmetric laminates, integration can be limited to from 0 to h/2).

TABLE 55 FORMULA FOR COUPLING MODULUS

	$\frac{1}{N^2} R_1$	$\frac{1}{N^2} R_2$
$\frac{2}{h^2} B_{11}$	$\sum F_t \cos 2\alpha_t$	$\sum F_t \cos 4\alpha_t$
$\frac{2}{h^2} B_{22}$	$-\sum F_t \cos 2\alpha_t$	$\sum F_t \cos 4\alpha_t$
$\frac{2}{h^2} B_{12}$		$-\sum F_t \cos 4\alpha_t$
$\frac{2}{h^2} B_{66}$		$-\sum F_t \cos 4\alpha_t$
$\frac{2}{h^2} B_{16}$	$-\frac{1}{2} \sum F_t \sin 2\alpha_t$	$-\sum F_t \sin 4\alpha_t$
$\frac{2}{h^2} B_{26}$	$-\frac{1}{2} \sum F_t \sin 2\alpha_t$	$\sum F_t \sin 4\alpha_t$

α_t = orientation of t-ply, $-N/2 \leq t \leq N/2$

$$I_{1B} = \frac{1}{4} (B_{11} + B_{22} + 2B_{12}) = 0 \tag{316}$$

$$I_{2B} = \frac{1}{8} (B_{11} + B_{22} - 2B_{12} + 4B_{66}) = 0 \tag{317}$$

$$R_{1B} = \frac{1}{2} \sqrt{(-B_{11} + B_{22})^2 + 4(B_{16} + B_{26})^2}$$

$$= \frac{R_1}{N^2} \sqrt{(\sum F_t \cos 2\alpha)^2 + (\sum F_t \sin 2\alpha)^2} = R_1 \sqrt{V_1^2 + V_3^2} \tag{318}$$

$$R_{2B} = \frac{1}{8} \sqrt{(B_{11} + B_{22} - 2B_{16} - 4B_{66})^2 + 16(B_{16} - B_{26})^2}$$

$$= \frac{R_2}{N^2} \sqrt{(\sum F_t \cos 4\alpha)^2 + (\sum F_t \sin 4\alpha)^2} = R_2 \sqrt{V_2^2 + V_4^2} \tag{319}$$

$$\tan 2\delta_1 = -\frac{2(B_{16}+B_{26})}{B_{11}-B_{22}} = -\frac{V_3}{V_1} \tag{320}$$

$$\tan 4\delta_1 = -\frac{4(B_{16}-B_{26})}{B_{11}+B_{22}-2B_{12}-4B_{66}} = -\frac{V_4}{V_2} \tag{321}$$

TABLE 56 FORMULA FOR TRANSFORMED COUPLING MODULUS

	$\sqrt{V_1^2+V_3^2}\quad R_1$	$\sqrt{V_2^2+V_4^2}\quad R_2$
$\frac{h^2}{2}B'_{11}$	$\cos 2(\theta-\delta_1)$	$\cos 4(\theta-\delta_2)$
$\frac{h^2}{2}B'_{22}$	$-\cos 2(\theta-\delta_1)$	$\cos 4(\theta-\delta_2)$
$\frac{h^2}{2}B'_{12}$	0	$-\cos 4(\theta-\delta_2)$
$\frac{h^2}{2}B'_{66}$	0	$-\cos 4(\theta-\delta_2)$
$\frac{h^2}{2}B'_{16}$	$-\frac{1}{2}\sin 2(\theta-\delta_1)$	$-\sin 4(\theta-\delta_2)$
$\frac{h^2}{2}B'_{26}$	$-\frac{1}{2}\sin 2(\theta-\delta_1)$	$\sin 4(\theta-\delta_2)$

$$V_1 = \frac{1}{N^2}\sum F_t \cos 2\alpha, \qquad V_2 = \frac{1}{N^2}\sum F_t \cos 4\alpha \tag{322}$$

$$V_3 = \frac{1}{N^2}\sum F_t \sin 2\alpha \qquad V_4 = \frac{1}{N^2}\sum F_t \sin 4\alpha \tag{323}$$

TABLE 57 CALCULATION OF REDUCTION FACTORS (V's) FOR COUPLING MODULUS

t	F_t	α_t	$\cos 2\alpha_t$	$F_t \cos 2\alpha_t$	$\sin 2\alpha_t$	$F_t \sin 2\alpha_t$	$\cos 4\alpha_t$	$F_t \cos 4\alpha_t$	$\sin 4\alpha_t$	$F_t \sin 4\alpha_t$
-7	-15									
-6	-13									
-5	-11									
-4	- 9									
-3	- 7									
-2	- 5									
-1	- 3									
0	- 1									
1	1									
2	3									
3	5									
4	7									
5	9									
6	11									
7	13									
8	15									

$h =$	$\Sigma Fc2 =$	$\Sigma Fs2 =$	$\Sigma Fc4 =$	$\Sigma Fs4 =$
$h_o =$	$\sqrt{V_1^2 + V_3^2} =$		$\sqrt{V_2^2 + V_4^2} =$	
$N =$	$-V_3 / V_1 =$		$-V_4 / V_2 =$	
$z_o =$	$\delta_1 =$		$\delta_2 =$	

TABLE 58 CALCULATION OF REDUCTION FACTORS (V's) FOR COUPLING MODULUS

t	F_t	α_t	$\cos 2\alpha_t$	$F_t \cos 2\alpha_t$	$\sin 2\alpha_t$	$F_t \sin 2\alpha_t$	$\cos 4\alpha_t$	$F_t \cos 4\alpha_t$	$\sin 4\alpha_t$	$F_t \sin 4\alpha_t$
-7	-15									
-6	-13									
-5	-11									
-4	- 9									
-3	- 7									
-2	- 5									
-1	- 3									
0	- 1									
1	1									
2	3									
3	5									
4	7									
5	9									
6	11									
7	13									
8	15									

$h =$ $\Sigma Fc2 =$ $\Sigma Fs2 =$ $\Sigma Fc4 =$ $\Sigma Fs4 =$

$h_o =$ $\sqrt{V_1^2 + V_3^2} =$ $\sqrt{V_2^2 + V_4^2} =$

$N =$ $- V_3 / V_1 =$ $- V_4 / V_2 =$

$z_o =$ $\delta_1 =$ $\delta_2 =$

TABLE 59 CALCULATION OF REDUCTION FACTORS (V's) FOR COUPLING MODULUS

		α_t	$\cos 2\alpha_t$	$\sin 2\alpha_t$	$\cos 4\alpha_t$	$\sin 4\alpha_t$

t	F_t	$F_t \cos 2\alpha_t$	$F_t \sin 2\alpha_t$	$F_t \cos 4\alpha_t$	$F_t \sin 4\alpha_t$
-7	-15				
-6	-13				
-5	-11				
-4	- 9				
-3	- 7				
-2	- 5				
-1	- 3				
0	- 1				
1	1				
2	3				
3	5				
4	7				
5	9				
6	11				
7	13				
8	15				

h =	$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$=	$\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =	
N =	$- V_3 / V_1$ =		$- V_4 / V_2$ =	
z_o =	δ_1 =		δ_2 =	

TABLE 60 CALCULATION OF REDUCTION FACTORS (V's) FOR COUPLING MODULUS

α_t	$\cos 2\alpha_t$		$\sin 2\alpha_t$		$\cos 4\alpha_t$		$\sin 4\alpha_t$	

t	F_t		$F_t \cos 2\alpha_t$	$F_t \sin 2\alpha_t$	$F_t \cos 4\alpha_t$	$F_t \sin 4\alpha_t$
-7	-15					
-6	-13					
-5	-11					
-4	-9					
-3	-7					
-2	-5					
-1	-3					
0	-1					
1	1					
2	3					
3	5					
4	7					
5	9					
6	11					
7	13					
8	15					

h =	$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$ =	$\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =	
N =	$-V_3 / V_1$ =		$-V_4 / V_2$ =	
z_o =	δ_1 =		δ_2 =	

TABLE 61 CALCULATION OF REDUCTION FACTORS (V's) FOR COUPLING MODULUS

t	F_t	α_t	$\cos 2\alpha_t$	$F_t \cos 2\alpha_t$	$\sin 2\alpha_t$	$F_t \sin 2\alpha_t$	$\cos 4\alpha_t$	$F_t \cos 4\alpha_t$	$\sin 4\alpha_t$	$F_t \sin 4\alpha_t$
-7	-15									
-6	-13									
-5	-11									
-4	- 9									
-3	- 7									
-2	- 5									
-1	- 3									
0	- 1									
1	1									
2	3									
3	5									
4	7									
5	9									
6	11									
7	13									
8	15									

h =	$\Sigma Fc2$ =	$\Sigma Fs2$ =	$\Sigma Fc4$ = $\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =		$\sqrt{V_2^2 + V_4^2}$ =
N =	$-V_3 / V_1$ =		$-V_4 / V_2$ =
z_o =	δ_1 =		δ_2 =

TABLE 62 CALCULATION OF REDUCTION FACTORS (V's) FOR COUPLING MODULUS

α_t	$\cos 2\alpha_t$	$\sin 2\alpha_t$	$\cos 4\alpha_t$	$\sin 4\alpha_t$

t	F_t	$F_t \cos 2\alpha_t$	$F_t \sin 2\alpha_t$	$F_t \cos 4\alpha_t$	$F_t \sin 4\alpha_t$
-7	-15				
-6	-13				
-5	-11				
-4	- 9				
-3	- 7				
-2	- 5				
-1	- 3				
0	- 1				
1	1				
2	3				
3	5				
4	7				
5	9				
6	11				
7	13				
8	15				

h =	$\Sigma Fc2$ = $\Sigma Fs2$ =	$\Sigma Fc4$ = $\Sigma Fs4$ =
h_o =	$\sqrt{V_1^2 + V_3^2}$ =	$\sqrt{V_2^2 + V_4^2}$ =
N =	$-V_3 / V_1$ =	$-V_4 / V_2$ =
z_o =	δ_1 =	δ_2 =

c. Computation of Reduction Factors, V's

(1) $[0_8/90_8]_T$ $\qquad h = 2mm$

$V_1 = -1/2; \qquad V_2 = V_3 = V_4 = 0$ (324)

$\delta_1 = 90$ (do not use 0 because B_{11} is negative)

For T-300/5208 $\quad R_1 = 85.85, \quad R_2 = 19.7$ GPa

$B_{11} = -B_{22} = \frac{h^2}{2} V_1 R_1 = -85.85$ kN (325)

$B_{12} = B_{55} = 0, \quad B_{16} = B_{26} = 0$ (326)

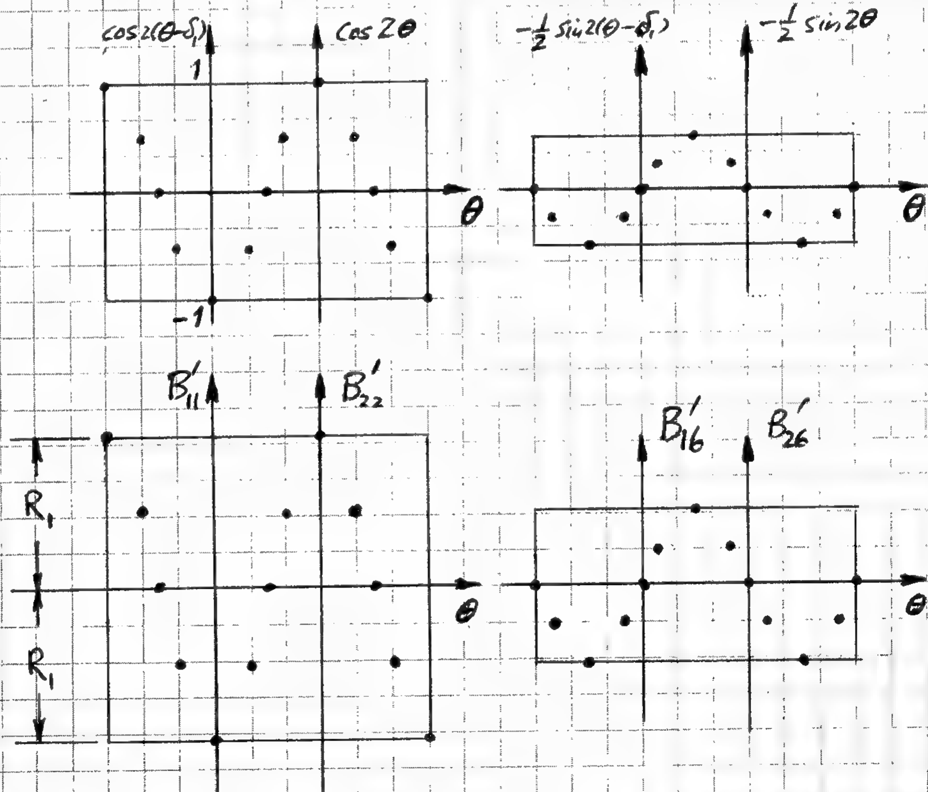


Figure 79 Transformed B_{ij} for $[0_8/90_8]$.

(2) $[45_8/-45_8]_T$

This is the same as $\theta = 45$ for $[0_8/90_8]_T$

For this laminate (angle-ply), $B_{11} = B_{22} = 0$, $B_{16} = -B_{26} = 42.9$ kN.

(3) $[0_4/90_4/0_4/90_4]_T$. B_{ij} for this laminate is exactly one half that of $[0_8/90_8]_T$. So for N - even, B_{ij}/N = constant.

3. COUPLED BENDING AND EXTENSION OF LAMINATES

a. Constitutive Relations

Between 2 generalized stresses (N_i & M_i), and 2 generalized deformations (e_i^o & k_i), there are 6 possible relations, all of which are listed below in terms of the original modulus matrices: Inplane A_{ij}, Coupling B_{ij} and Flexural D_{ij}.

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} e^o \\ k \end{Bmatrix} \qquad \begin{aligned} a &= A^{-1} \\ b &= B^{-1} \\ d &= D^{-1} \end{aligned} \tag{327}$$

$$\begin{Bmatrix} e^o \\ M \end{Bmatrix} = \begin{bmatrix} a & -aB \\ Ba & D-BaB \end{bmatrix} \begin{Bmatrix} N \\ k \end{Bmatrix} \qquad \begin{Bmatrix} N \\ k \end{Bmatrix} = \begin{bmatrix} A-BdB & Bd \\ -dB & d \end{bmatrix} \begin{Bmatrix} e^o \\ M \end{Bmatrix} \tag{328}$$

$$\begin{Bmatrix} N \\ e^o \end{Bmatrix} = \begin{bmatrix} B-AbD & Ab \\ -bD & b \end{bmatrix} \begin{Bmatrix} k \\ M \end{Bmatrix} \qquad \begin{Bmatrix} k \\ M \end{Bmatrix} = \begin{bmatrix} b & -bA \\ Db & B-DbA \end{bmatrix} \begin{Bmatrix} N \\ e^o \end{Bmatrix} \tag{329}$$

$$\begin{Bmatrix} e^o \\ k \end{Bmatrix} = \begin{bmatrix} (A-BdB)^{-1} & -(A-BdB)^{-1}Bd \\ -(D-BaB)^{-1}Ba & (D-BaB)^{-1} \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{330}$$

All submatrices are 3 x 3, which are numerically simple to calculate. The unpartitioned, original matrices are 6 x 6, which normally require bigger computer to evaluate. A sample problem will be solved in the next section using the 3 x 3 submatrices only.

b. Numerical Example

T-300/5208 $\quad [0_8/90_8]_T$, $\quad h = 2\text{mm}$

$$Q_{ij} = \begin{bmatrix} 182 & 2.9 & 0 \\ & 10.3 & 0 \\ & & 7.2 \end{bmatrix} \text{GPa} \tag{331}$$

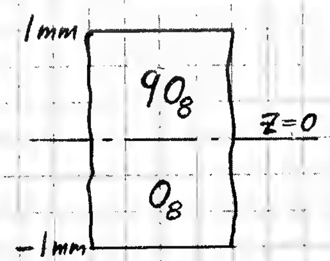


Figure 80 2-layer unsymmetric laminate.

$$A_{ij} = \begin{bmatrix} 192.3 & 5.8 & 0 \\ & 192.3 & 0 \\ & & 14.4 \end{bmatrix} \quad MNm^{-1} \qquad (332)$$

$$B_{ij} = \begin{bmatrix} -85.85 & 0 & 0 \\ & 85.85 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad kN \qquad (333) \qquad D_{ij} = \begin{bmatrix} 64.1 & 1.93 & 0 \\ & 64.1 & 0 \\ & & 4.8 \end{bmatrix} \quad Nm \qquad (334)$$

$$|A| = 532017 \qquad a_{ij} = \begin{bmatrix} 5.20 & -.157 & 0 \\ & 5.20 & 0 \\ & & 69.4 \end{bmatrix} \quad (GNm^{-1})^{-1} \qquad (335)$$

$|B| = 0$, B_{ij} is singular, therefore, there is no inverse.

$$|D| = 19740 \qquad d_{ij} = \begin{bmatrix} 15.58 & -.469 & 0 \\ & 15.58 & 0 \\ & & 208 \end{bmatrix} \quad (kNm)^{-1} \qquad (336)$$

$$Ba = \begin{bmatrix} -446 & 13.48 & 0 \\ -13.48 & 446 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (Mm^{-1})^{-1} \qquad (337)$$

$$aB = \begin{bmatrix} -446 & -13.48 & 0 \\ 13.48 & 446 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (Mm^{-1})^{-1} \qquad (338)$$

$$BaB = \begin{bmatrix} 38.3 & 1.16 & 0 \\ & 38.3 & 0 \\ & & 0 \end{bmatrix} \quad Nm \qquad (339) \qquad D-BaB = \begin{bmatrix} 25.8 & .77 & 0 \\ & 25.8 & 0 \\ & & 4.8 \end{bmatrix} \quad Nm \qquad (340)$$

$$Bd = \begin{bmatrix} -1337 & 40.26 & 0 \\ -40.26 & 1337 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad m^{-1} \qquad (341)$$

$$dB = \begin{bmatrix} -1337 & -40.26 & 0 \\ 40.26 & 1337 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; m^{-1} \qquad (342)$$

$$BdB = \begin{bmatrix} 114 & 3.46 & 0 \\ & 114 & 0 \\ & & 0 \end{bmatrix} \; MNm^{-1} \qquad (343)$$

$$A-BdB = \begin{bmatrix} 78.3 & 2.34 & 0 \\ & 78.3 & 0 \\ & & 14.4 \end{bmatrix} \; MNm^{-1} \qquad (344)$$

$$(D-BaB)^{-1} = \begin{bmatrix} 38.8 & -1.16 & 0 \\ & 38.8 & 0 \\ & & 208 \end{bmatrix} \; (kNm)^{-1} \qquad (345)$$

$$(A-BdB)^{-1} = \begin{bmatrix} 12.8 & -.382 & 0 \\ & 12.8 & 0 \\ & & 69.4 \end{bmatrix} \; (GNm^{-1})^{-1} \qquad (346)$$

$$(D-BaB)^{-1}Ba = \begin{bmatrix} -17.29 & 0 & 0 \\ 0 & 17.29 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; (MN)^{-1} \qquad (347)$$

$$(A-BdB)^{-1}Bd = \begin{bmatrix} -17.13 & 0 & 0 \\ 0 & 17.13 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; (MN)^{-1} \qquad (348)$$

c. Summary of Constitutive Equations for T-300/5208 $[0_8/90_8]_T$, $h = 2$mm

Units: - N is MNm^{-1}

M is N

ϵ^o is 10^{-3}m/m or mm/m

k is m^{-1}

$$\begin{Bmatrix} N \\ \\ M \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 192.3 & 5.8 & 0 & -85.85 & 0 & 0 \\ 5.8 & 192.3 & 0 & 0 & 85.85 & 0 \\ 0 & 0 & 14.4 & 0 & 0 & 0 \\ \hline -85.85 & 0 & 0 & 64.1 & 1.93 & 0 \\ 0 & 85.85 & 0 & 1.93 & 64.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.8 \end{array}\right] \begin{Bmatrix} \epsilon^o \\ \\ k \end{Bmatrix} \qquad (349)$$

Units: $\left[\begin{array}{c|c} MNm^{-1} & kN \\ \hline kN & Nm \end{array}\right]$

$$\begin{Bmatrix} \epsilon^o \\ \\ k \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 12.8 & -.382 & 0 & 17 & 0 & 0 \\ -.382 & 12.8 & 0 & 0 & -17 & 0 \\ 0 & 0 & 69.4 & 0 & 0 & 0 \\ \hline 17 & 0 & 0 & 38.8 & -1.16 & 0 \\ 0 & -17 & 0 & -1.16 & 38.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 208 \end{array}\right] \begin{Bmatrix} N \\ \\ M \end{Bmatrix} \qquad (350)$$

Units : $\left[\begin{array}{c|c} (GNm^{-1})^{-1} & (MN)^{-1} \\ \hline (MN)^{-1} & (kNm)^{-1} \end{array}\right] \qquad (351)$

(1) Determine $N_{1(FPF)}$ under uniaxial tension.

$$\epsilon_1^o = 12.8\, N_1 \quad , \quad k_1 = 17 N_1 \quad , \qquad (352)$$

$$\therefore \epsilon_1\Big]_{z=1mm} = (12.8 + 1 \times 17) N_1 = 29.8\, N_1 \qquad (353)$$

Let $e_{FPF} = 3.91$ mm/m (354)

$$\therefore N_{1(FPF)} = \frac{3.91}{29.8} = .131 \text{ MNm}^{-1} \tag{355}$$

$$\bar{\sigma}_{1(FPF)} = N_{1(FPF)}/h = 65.6 \text{ MPa} \tag{356}$$

(2) Determine $N_{1(FPF)}$ if the same plies were symmetrical;

e.g., instead of $[0_8/90_8]_T$, use $[0_4/90_4]_s$.

For the latter laminate,

A_{ij} remains same, but $B_{ij} = 0$.

$$e_1 = 12.8N_1, \quad k_1 = 0. \tag{357}$$

$$N_{1(FPF)} = \frac{3.91}{12.8} = .305 \text{ MNm}^{+1} \tag{358}$$

$$\bar{\sigma}_{1(FPF)} = 153 \text{ MPa} \tag{359}$$

(About 2.3 times higher than the unsymmetric laminate)

d. Alternative Constitutive Equations of T-300/5208 $[0_8/90_8]_T$ Laminate (From Equation 328)

$$\begin{Bmatrix} \epsilon^o \\ \\ \\ \\ M \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 5.20 & -.157 & 0 & 446 & 13.48 & 0 \\ -.157 & 5.20 & 0 & -13.48 & -446 & 0 \\ 0 & 0 & 69.4 & 0 & 0 & 0 \\ \hline -446 & 13.48 & 0 & 25.8 & .77 & 0 \\ -13.48 & 446 & 0 & .77 & 25.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.8 \end{array}\right] \begin{Bmatrix} N \\ \\ \\ \\ k \end{Bmatrix} \tag{360}$$

$$\text{Units :} \quad \left[\begin{array}{c|c} (\mathrm{GNm}^{-1})^{-1} & (\mathrm{Mm}^{-1})^{-1} \\ \hline (\mathrm{Mm}^{-1})^{-1} & \mathrm{Nm} \end{array}\right] \tag{361}$$

$$\begin{Bmatrix} N \\ \\ \\ \\ k \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 78.3 & 2.34 & 0 & -1337 & 40.26 & 0 \\ 2.34 & 78.3 & 0 & -40.26 & 1337 & 0 \\ 0 & 0 & 14.4 & 0 & 0 & 0 \\ \hline 1337 & 40.26 & 0 & 15.58 & -.469 & 0 \\ -40.26 & -1337 & 0 & -.469 & 15.58 & 0 \\ 0 & 0 & 0 & 0 & 0 & 208 \end{array}\right] \begin{Bmatrix} \epsilon^o \\ \\ \\ \\ M \end{Bmatrix} \tag{362}$$

$$\text{Units :} \quad \left[\begin{array}{c|c} \mathrm{MNm}^{-1} & \mathrm{m}^{-1} \\ \hline \mathrm{m}^{-1} & (\mathrm{kNm})^{-1} \end{array}\right] \tag{363}$$

(1) Determine ply stresses in uniaxial extension (N_1 only) of a tubular specimen of $[0_8/90_8]$ ply orientation.

For long cylindrical tubes: $k_1 = k_2 = k_6 = 0$ (364)

$$\epsilon_1^o = 5.20 N_1 \quad (365)$$

$$\epsilon_2^o = -.157 N_1 \quad (366)$$

$$M_1 = 446 N_1 \quad (367)$$

$$M_2 = 13.48 N_1 \quad (368)$$

Moments are induced because curvature is prevented.

At FPF -

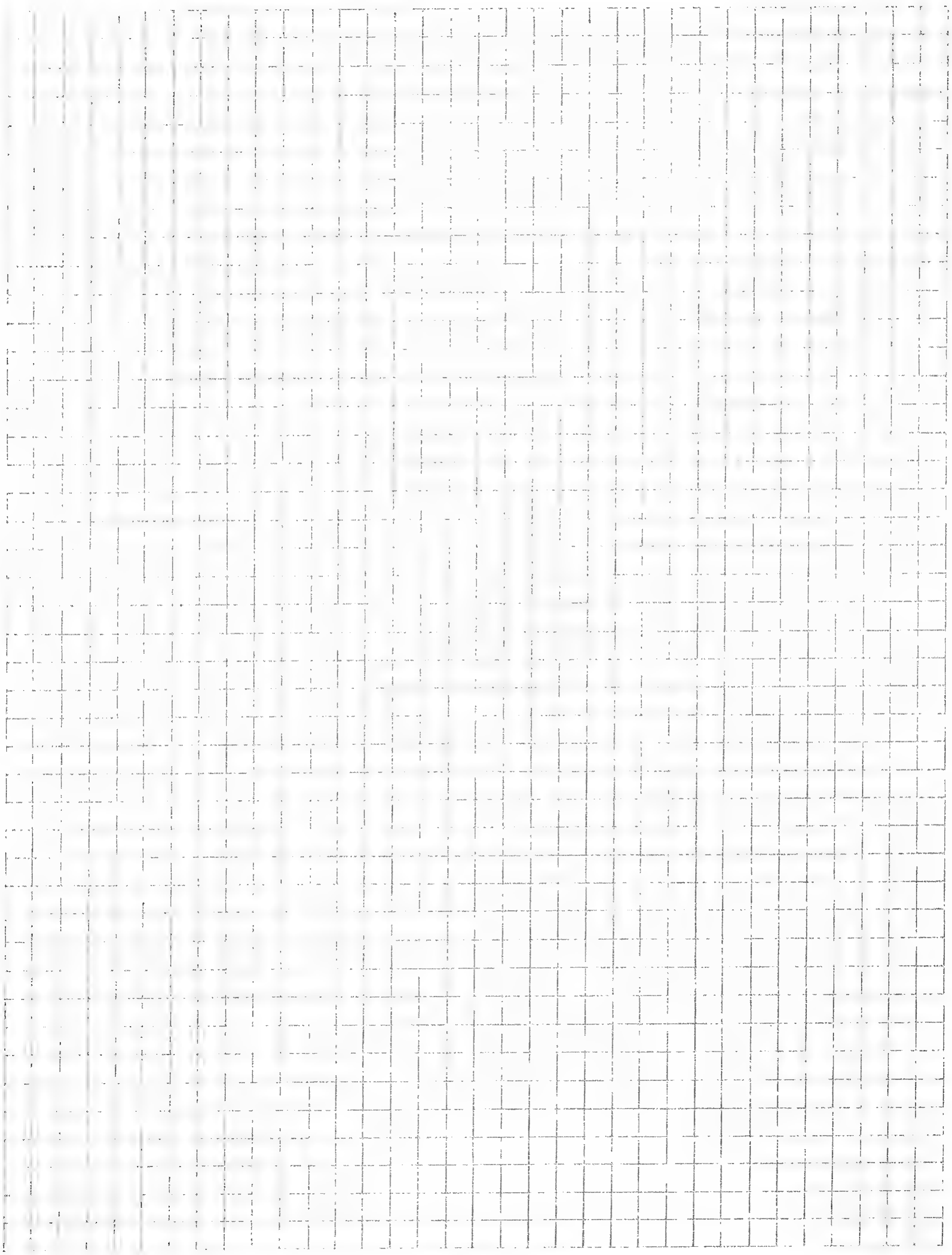
$$N_{1(FPF)} = \frac{3.91}{5.20} = .752 \text{ MNm}^{-1} \quad (369)$$

$$\bar{\sigma}_{1(FPF)} = 375 \text{ MPa} \quad (370)$$

The tube has higher FPF-stress than both unsymmetric (65.6) and symmetric (153) laminates. The induced moments have beneficial effects.

(2) Are they still beneficial if the tube is under axial compression?

(3) What is the difference between shear response of flat laminates and tubes?



SECTION VIII

STRESS AND DEFORMATION DUE TO CURING AND SWELLING

1. UNIDIRECTIONAL LAMINAE

a. Moisture Concentration

Dry State

$$M = M_m + M_f \tag{371}$$

$$V = V_m + V_f + V_v \tag{372}$$

Wet State

$$M' = M + M_{mw} + M_{fw} + M_{vw} \tag{373}$$

Moisture concentration

$$c = \frac{M' - M}{M} = c_m m_m + c_f m_f + M_{vw} / M$$

$$= (c_m v_m \rho_m + c_f v_f \rho_f + v_v \rho_w) / \rho \tag{374}$$

$$= (c_m v_m s_m + c_f v_f s_f + v_v) / s$$

V_v : volume of voids

M_{mw} : mass of water in matrix

M_{fw} : " " " fiber

M_{vw} " " " voids

$$c_{[f, m]} = M_{[f, m]w} / M_{[f, m]} \tag{375}$$

s : specific gravity

In many cases $c_f \approx 0$

$$c = (c_m v_m s_m + v_v) / s \tag{376}$$

(1) Sample Problem

Maximum water absorption in a typical epoxy is 6%.

What is the maximum amount of water which can be absorbed in a Gr/Ep ($v_f = 0.65$)? Specific gravities for the epoxy and composite are 1.25 and 1.6, respectively, and the void content is 0.4%.

Solution

Since $c_f = 0$, $c = (c_m v_m s_m + v_v) / s = 1.89\%$

b. Approximations for Curing and Swelling Strains

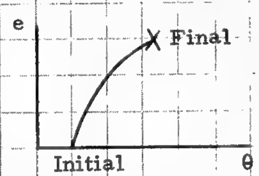
$$e_L = \frac{v_f E_f e_f + v_m E_m e_m}{\eta_l v_f E_f + v_m E_m} \quad : \quad \text{longitudinal strain} \qquad (377)$$

$$e_T = v_f e_f + v_m e_m + v_f \nu_f e_f + v_m \nu_m e_m$$

$$- (\eta_I v_f \nu_f + v_m \nu_m) e_L \quad : \quad \text{transverse strain} \qquad (378)$$

N = T (curing), or H (swelling), or T + H (both)

$e_{[f,m]} = e^T_{[f,m]} + e^H_{[f,m]}$: strains resulting from a change of temperature and moisture concentration, measured from the initial, stress-free state to a final state.



$E_{[f,m]}$, $\nu_{[f,m]}$: Elastic constants at the final state of interest.

Curing strains

$$e^T_{[f,m]} \approx \alpha^T_{[f,m]} \Delta T \qquad (379)$$

ΔT : (temperature of interest) - (curing temperature)

Swelling strains

$$e^H_f \approx 0 \qquad (380)$$

$$c_f \approx 0 \qquad (381)$$

$$e^H_m \approx \alpha^H_m c_m \qquad (382)$$

$$\alpha_m^H = \frac{1}{3} s_m \quad ; \quad \text{volume additivity} \tag{383}$$

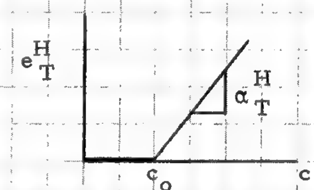
$$e_L^H \approx 0 \tag{384}$$

$$e_T^H \approx \frac{1+\nu_m}{3} v_m s_m c_m$$

$$= \frac{1+\nu_m}{3} s\,(c - c_o) \tag{385}$$

In general

$$e_T^H = \alpha_T^H \,(c - c_o)\, H\,(c_o) \tag{386}$$



α_T^H : Swelling coefficient

H( ) : Heaviside step function

c_o : $= v_v/s$; in general, threshhold value of c_o below which no appreciable swelling occurs.

(1) Sample Problem

What is the expected swelling coefficient of the Gr/Ep in sample problem 1.a.(1)? Poisson's ratio of a typical epoxy is 0.35.

$$c_o = v_v/s = 0.25\%$$

$$\alpha_T^H = \frac{1+\nu_m}{3} s = 0.72$$

c. Approximations for Residual Stresses in Unidirectional Laminae

Refer to Section IV.

Matrix

$$\bar{\sigma}^R_{mL} = \frac{v_f E_f E_m (e_f - \eta_1 e_m)}{\eta_1 v_f E_f + v_m E_m} \tag{387}$$

$$\bar{\sigma}^R_{mT} = 0 \tag{388}$$

Fiber

$$\bar{\sigma}^R_{fL} = -\frac{v_m}{v_f}\,\bar{\sigma}^R_{mL} = \frac{v_m E_m E_f (\eta_1 e_m - e_f)}{\eta_1 v_f E_f + v_m E_m} \tag{389}$$

$$\bar{\sigma}^R_{fT} = 0 \tag{390}$$

Residual stresses manifest themselves in the residual fringes in photoelastic composites.

d. Change of Densities Due to Curing

Volume changes

$$\frac{\Delta V_m}{V_m} = \frac{e_L}{\eta_1 v_f + v_m} + 2\left[e_m + \frac{\eta_1 v_f E_f \nu_m (e_m - e_f/\eta_1)}{\eta_1 v_f E_f + v_m E_m}\right] \tag{391}$$

$$\frac{\Delta V_f}{V_f} = \frac{\eta_1 e_L}{\eta_1 v_f + v_m} + 2\left[e_f + \frac{\eta_1 v_m E_m \nu_f (e_f/\eta_1 - e_m)}{\eta_1 v_f E_f + v_m E_m}\right] \tag{392}$$

In-situ densities (dry) after curing

$$\eta_1 = 1$$

$$\rho_m = \rho_{mo}\,\frac{1 + 3e^T_m}{1 + 2e^T_m\,(1 + \nu_m) + (1 - 2\nu_m)e^T_L} \approx \rho_{mo} \tag{393}$$

For Gr/Ep, $\nu_m = 0.35$, $e^L_m = 0$, $-e^T_m \approx 0.77\%$.

$$\rho_m = \rho_{mo}\,\frac{1 + 3e^T_m}{1 + 2.7e^T_m} = 0.9976\,\rho_{mo} \approx \rho_{mo}$$

$$\rho_f = \rho_{fo} \frac{1+3e_f^T}{1+2e_f^T(1+\nu_f)+(1-2\nu_f)e_L^T} \approx \rho_{fo} \quad (394)$$

ρ_{mo}, ρ_{fo} : densities of constituents themselves at the temperature of interest

Composite density

$$M = M_m + M_f \quad (395)$$

$$V = M_m/\rho_m + M_f/\rho_f + V_v \quad (396)$$

$$\rho = \rho_m v_m + \rho_f v_f \quad (397)$$

e. Change of Densities Due to Moisture Absorption

Dry State

$$M = M_m + M_f, \quad V = V_m + V_f + V_v \quad (398)$$

Wet State

$$M' = M + M_{mw} + M_{fw} + M_{vw} \quad (399)$$

$$V' = V + \Delta V_m + \Delta V_f \quad (400)$$

Wet density

$$\rho' = \frac{M'}{V'} = \frac{M'}{M}\frac{M}{V}\frac{V}{V'} = \frac{\rho(1+c)}{1+\Delta V_m/V+\Delta V_f/V}$$

$$\approx \rho \frac{1+c}{1+2v_m(1+\nu_m)e_m^H}$$

$$\approx \rho \frac{1+c}{1+2v_m(1+\nu_m)s_m c_m/3} \quad (401)$$

or

$$\frac{V}{V'} = \frac{1}{1+c}\frac{\rho'}{\rho} \quad (402)$$

f. Remarks on Thermal Expansion Coefficients of Composites

Assume

$$e^T_{[f,m]} = \alpha^T_{[f,m]}(T - T_o) \tag{403}$$

$\alpha^T_{[f,m]}$: independent of T

T_o : initial reference temperature

From Eq. (377)

$$\alpha^T_L(T) = \frac{v_f E_f(T)\alpha^T_f + v_m E_m(T)\alpha^T_m}{\eta_1 v_f E_f(T) + v_m E_m(T)} \tag{404}$$

where

$$\alpha^T_L(T) = \frac{e^T_L(T)}{T - T_o} \tag{405}$$

Comments

1. T_o is not arbitrary and e^T_L must be measured from T_o.

2. α^T_L depends on T while α^T_f and α^T_m do not.

Define

$$\alpha^T_L(T_2, T_1) = \frac{e^T_L(T_2) - e^T_L(T_1)}{T_2 - T_1} \tag{406}$$

From Eq. (377)

$$\alpha^T_L(T_2, T_1) = \frac{v_f E_f(T_2)\alpha^T_f + v_m E_m(T_2)\alpha^T_m}{\eta_1 v_f E_f(T_2) + v_m E_m(T_2)} + \left(\frac{v_f E_f(T_2)\alpha^T_f + v_m E_m(T_2)\alpha^T_m}{\eta_1 v_f E_f(T_2) + v_m E_m(T_2)} - \frac{v_f E_f(T_1)\alpha^T_f + v_m E_m(T_1)\alpha^T_m}{\eta_1 v_f E_f(T_1) + v_m E_m(T_1)}\right) \frac{T_1 - T_o}{T_2 - T_1} \tag{407}$$

$\alpha^T_L(T_2, T_1)$ becomes independent of temperature if so are the moduli.

2. MULTIDIRECTIONAL LAMINATES

a. Curing and Swelling Strains

Constitutive relations of a constituent ply

$$\sigma_i = \mathcal{C}_{ij}(\epsilon_j - e_j) + \mathcal{C}_{iA}(\epsilon_A - e_A) \quad : \text{ in-plane stresses} \tag{408}$$

$$\sigma_A = \mathcal{C}_{Aj}(\epsilon_j - e_j) + \mathcal{C}_{AB}(\epsilon_B - e_B) \quad : \text{ out-of-plane stresses} \tag{409}$$

Reduced stiffnesses Q_{ij}

$$Q_{ij} = \mathcal{C}_{ij} - \mathcal{C}_{iA}\mathcal{C}^{-1}_{AB}\mathcal{C}_{Bj} \tag{410}$$

$$\sigma_i = Q_{ij}(\epsilon_j - e_j) + \mathcal{C}_{iA}\mathcal{C}^{-1}_{AB}\sigma_B \tag{411}$$

In-plane strains and curvatures

$$\epsilon_i = \epsilon^o_j + z k_j \tag{412}$$

Classical laminated plate theory

σ_B, ϵ^o_j, k_j independent of z

$$\int \sigma_i dz = N_i = A_{ij}\epsilon^o_j + B_{ij}k_j - N^N_i + T_{iB}\sigma_B \tag{413}$$

$$\int \sigma_i z dz = M_i = B_{ij}\epsilon^o_j + D_{ij}k_j - M^N_i + R_{iB}\sigma_B \tag{414}$$

$$\frac{1}{h}\int \epsilon_A dz = \overline{\epsilon}_A = \overline{\mathcal{C}^{-1}_{AB}}\,\sigma_B - \overline{\mathcal{C}^{-1}_{AB}\mathcal{C}_{Bj}}\,\epsilon^o_j - \overline{\mathcal{C}^{-1}_{AB}\mathcal{C}_{Bj}z}\,k_j + \overline{\mathcal{C}^{-1}_{AB}\mathcal{C}_{Bj}e_j} + \overline{e_A} \tag{415}$$

where

$$[N^N_i, M^N_i] = \int_{-h/2}^{h/2} Q_{ij}e_j\,[1, z]dz \tag{416}$$

$$[T_{iB}, R_{iB}] = \int_{-h/2}^{h/2} \mathcal{C}_{iA}\mathcal{C}^{-1}_{AB}\,[1, z]dz \tag{417}$$

$$\overline{(\quad)} = \frac{1}{h}\int_{-h/2}^{h/2} (\quad)dz \tag{418}$$

Nonmechanical laminate strains and curvatures

$$N_i = M_i = \sigma_A = 0 \tag{419}$$

$$e_i^o = F_{ij}^{-1} (N_j^N - B_{jk} D_{kn}^{-1} M_n^N) \tag{420}$$

$$k_i^N = D_{ij}^{-1} (M_j^N - B_{jk} e_k^o) \tag{421}$$

$$\overline{e}_A = \overline{e}_A + \overline{C_{AB}^{-1} C_{Bj} e_j} - \overline{C_{AB}^{-1} C_{Bj}} e_j^o - \overline{C_{AB}^{-1} C_{Bj} z} \quad k_j^N \tag{422}$$

where

$$F_{ij} = A_{ij} - B_{ik} D_{kn}^{-1} B_{nj} \tag{423}$$

Strain-displacement relations

$$e_1^o = \partial u/\partial x_1, \quad e_2^o = \partial v/\partial x_2, \quad e_6^o = \partial u/\partial x_2 + \partial v/\partial x_1 \tag{424}$$

$$k_1 = -\partial^2 w/\partial x_1^2, \quad k_2 = -\partial^2 w/\partial x_2^2, \quad k_6 = -2\partial^2 w/\partial x_1 \partial x_2 \tag{425}$$

All the elastic constants are taken at the final state (T, H).

In the material symmetry axes of transversely isotropic laminae

$$Q_{ij} = \begin{bmatrix} E_L/U_1 & E_T \nu_{LT}/U_1 & 0 \\ & E_T/U_1 & 0 \\ & & G_{LT} \end{bmatrix} \tag{426}$$

$$C_{AB}^{-1} = \begin{bmatrix} 1/E_T & 0 & 0 \\ & 2(1+\nu_{TT})/E_T & 0 \\ & & 1/G_{LT} \end{bmatrix} \tag{427}$$

$$C_{iA} = C_{Ai}^{T} = \begin{bmatrix} \nu_{LT}(1+\nu_{TT})E_T/U_2 & 0 & 0 \\ (\nu_{TT}+\nu_{LT}\nu_{TL})E_T/U_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (428)$$

$$U_1 = 1 - \nu_{LT}\nu_{TL} \quad (429)$$

$$U_2 = (1+\nu_{TT})(1-\nu_{TT} - 2\nu_{LT}\nu_{TL}) \quad (430)$$

b. Curing Strains and Residual Stresses of Symmetric Laminates

$$B_{ij} = 0 , \qquad M_i^N = 0 , \qquad k_i^N = 0 \quad (431)$$

TABLE 63. LAMINA CURING STRAINS FROM CURING TEMPERATURE [12]

	V_f	s	e_L^T %	e_T^T %	T_o ^oK
Boron/Epoxy (B/Ep)	0.50	2.03	0.118	0.451	450
Boron/Polyimide (B/PI)	0.49	2.00	0.081	0.443	450
Graphite/Epoxy (Gr/Ep)	0.45	1.54	-0.009	0.318	445
Graphite/Polyimide (Gr/PI)	0.45	1.54	0.	0.391	450
Glass/Epoxy (Gl/Ep)	0.72	2.13	0.072	0.530	435

TABLE 64 LAMINA MECHANICAL PROPERTIES [12]

	E_L GNm^{-2}	E_T GNm^{-2}	G_{LT} GNm^{-2}	ν_{LT}	X MNm^{-2}	Y MNm^{-2}	S MNm^{-2}
B/Ep	201	21.7	5.4	0.17	1375	56.0	62.3
B/PI	222	14.5	7.7	0.16	1040	10.8	25.9
Gr/Ep	190	7.10	6.2	0.10	1115	41.9	61.5
Gr/PI	216	4.97	4.5	0.25	841	14.9	21.7
Gl/Ep	60.7	24.8	12.0	0.23	807	46.0	45.0

TABLE 65 CURING STRAINS OF $[0_2/\pm45]$ LAMINATES AND CORRESPONDING STRESSES WITHIN 45° PLY [12,19]

	$-e_0^T$, %		$-e_{90}^T$, %		$-e_{45}^T$, %		σ_T^R	σ_{LT}^R
	EXP.	CAL.	EXP.	CAL.	EXP.	CAL.	Y	S
B/Ep	0.123	0.109	0.234	0.259	0.171	0.184	0.99	0.13
B/PI	0.100	0.075	0.214	0.186	0.163	0.131	4.09	0.33
Gr/Ep	-0.018 (-0.009)	-0.019	0.104 (0.068)	0.122	0.036 (0.026)	0.052	1.09	0.11
Gr/PI	0.002	-0.004	0.031	0.051	0.017	0.028	1.20	0.11
Gl/Ep	0.105	0.117	0.260	0.361	0.182	0.239	1.39	0.65

Remarks: Numbers inside the parentheses for Gr/Ep are measured during the cooling stage of curing and the heating stage of postcuring.

c. Warping of Asymmetric Laminates After Curing

Input mechanical properties at room temperature

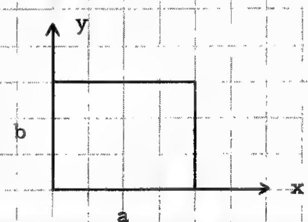
Material	E_L	E_T	G_{LT}	ν_{LT}
	GPa	GPa	GPa	
Gr/Ep	153	11.2	7.10	0.33
Gl/Ep	39.1	13.1	4.70	0.30

Input curing strains for lamina

Material	$T_o = 450^\circ K$		$T_o = 394^\circ K$	
	e_L^T, mm/m	e_T^T, mm/m	e_L^T, mm/m	e_T^T, mm/m
Gr/Ep	0.304	- 3.503	0.267	- 2.167
Gl/Ep	- 1.345	- 4.077	- 0.743	- 2.235

Amount of warping

$$w^T = -\frac{1}{2}(k_1^T x^2 + k_2^T y^2 + k_6^T xy) + c_1 x + c_2 y + c_3 \quad (432)$$



(433)

For $[+\theta/-\theta]$ laminates

$$k_1^T = k_2^T = 0 \quad , \quad w_{max}^T(a, b) = -\frac{k_6^T}{2}ab \quad (434)$$

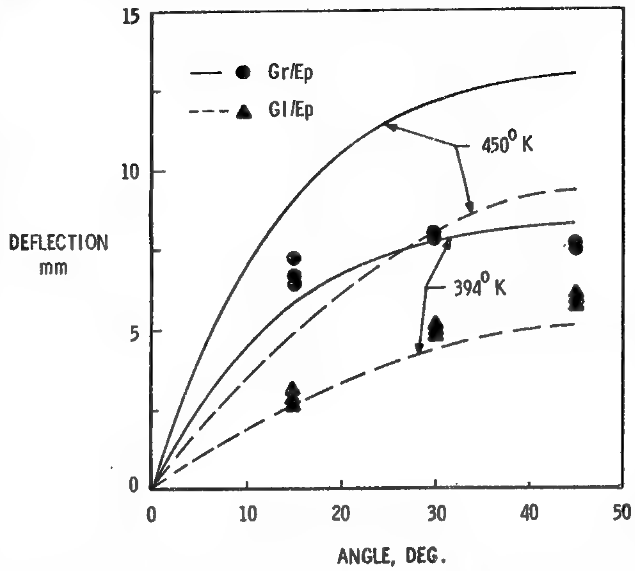


Figure 81 Maximum deflection of $[\theta_4/-\theta_4]_T$ laminates after curing, a = b = 6.35cm. [20]

d. Swelling Strains of Symmetric Laminates

$$k_i^H = 0 \tag{435}$$

$$e_T^H = \alpha_T^H (c-c_o) H(c_o), \quad e_L^H = 0 \tag{436}$$

$$\frac{c-c_o}{c_\infty-c_o} = 1 - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \cos \frac{(2j+1)\pi z}{h} \exp - \frac{(2j+1)^2 \pi^2 dt}{h^2} \tag{437}$$

$$\frac{\overline{c}-c_o}{c_\infty-c_o} = 1 - \frac{8}{\pi^2} \sum_{j=o}^{\infty} \frac{1}{(2j+1)^2} \exp - \frac{(2j+1)^2 \pi^2 Dt}{h^2} \tag{438}$$

Absorption

$$c_o = 0 \tag{439}$$

Desorption

$$c_\infty = 0 \tag{440}$$

Properties of Gr/Ep (AS/3501) lamina

E_L, GPa	E_T, GPa	G_{LT}, GPa	ν_{LT}	ν_{TT}	e_L^T, mm/m	e_T^T, mm/m
153	11.2	7.10	0.33	0.33	0.127	-4.228

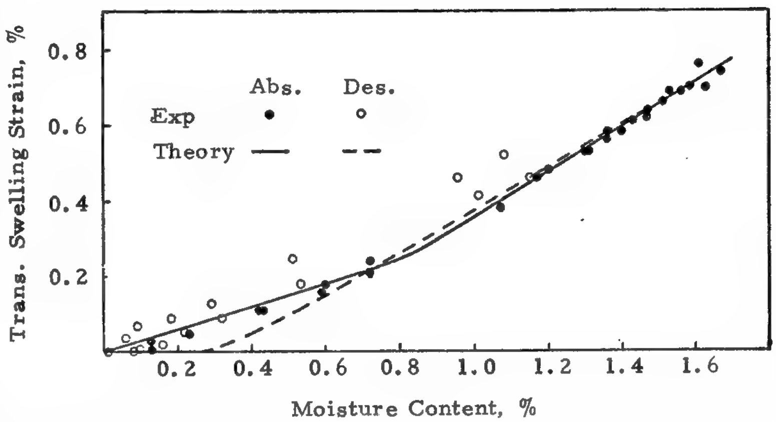


Figure 82 Transverse swelling strain in [0] Gr/Ep (AS/3501) subjected to absorption and desorption tests. Note that

$\alpha_T^H = 0.59$, $c_o = 0.4\%$.

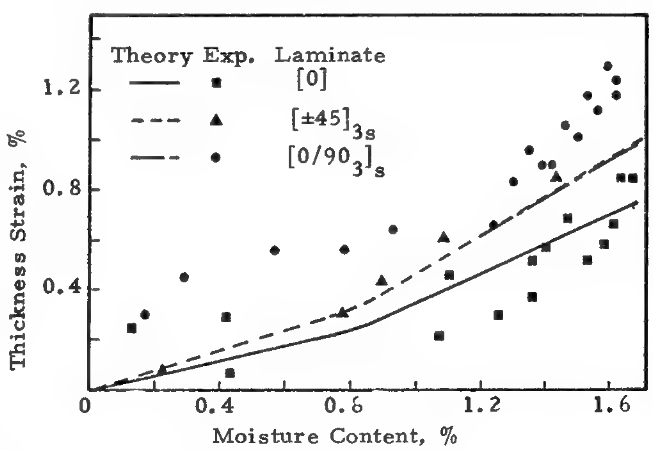


Figure 83 Swelling strain in thickness direction of Gr/Ep laminates.

e. Warping of Asymmetric Laminates Due to Swelling

$$\text{Total } w^N = w^T + w^H \tag{441}$$

For square [0/90] laminate of side a, $k_6^N = 0$, $k_1^N = k_2^N$.

$$w_{max}^N = -\frac{1}{2}\left(\frac{a}{2}\right)^2 (k_1^T + k_1^H) \tag{442}$$

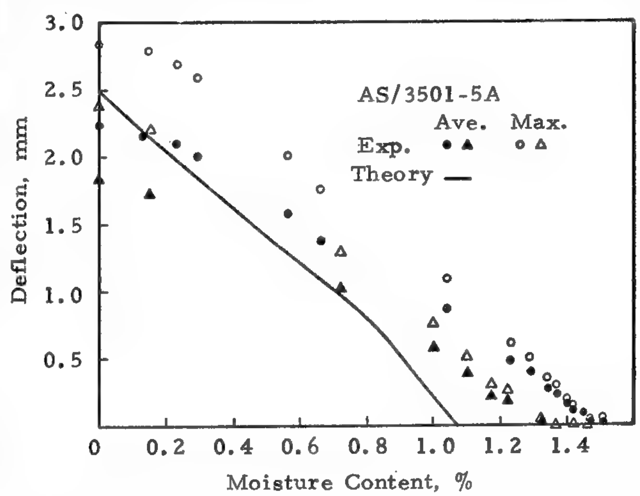


Figure 84 Maximum deflection of $[0_4/90_4]_T$ Gr/Ep (AS/3501) laminate due to curing and moisture absorption, a = 7.58cm.

SECTION IX

FAILURE THEORIES

1. INTRODUCTION

Definition of failure:

Any consistently definable point on the stress-strain relation

Simplifying Hypothesis:

Only one point of failure associated with one state of stress

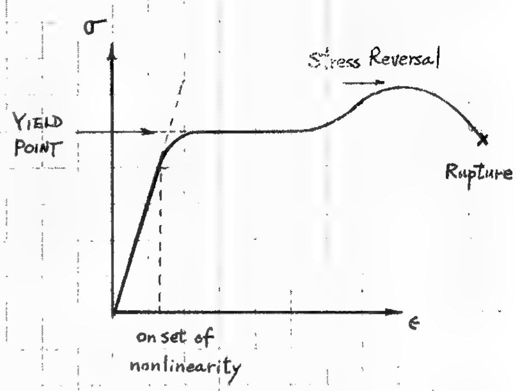
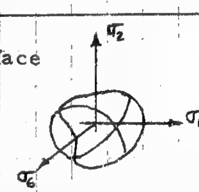


Figure 85 Definition of failure.

Geometric interpretation of failure (in stress space)

Dimension of Physical Space	Dimension of Stress Space	Failure Contour
One (x_1)	One (σ_1)	Points $-\sigma_1$ X_1' O X_1 σ_1
Two (x_1, x_2)	Three ($\sigma_1, \sigma_2, \sigma_6$)	3-D Surface 
Three (x_1, x_2, x_3)	Six ($\sigma_1, \sigma_2 \cdots \sigma_6$)	6-D hypersurface

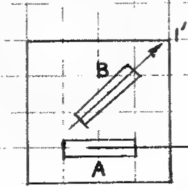
Failure Criterion: Definition of the loci (or contour) of failure surface.

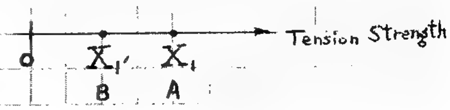
Representation: Material is safe for any state of stress within the failure surface.

Application: Predictive design.

Strength Anisotropy - Failure surface dependent on material orientation

1-D example





2. GEOMETRIC INTERPRETATION OF EMPLOYING FAILURE CRITERION TO INTERROGATE POTENTIAL FAILURE

A given state of stress σ_1, σ_2, σ_6 (in 2-D) can be represented as a vector $\mathcal{S}$ in stress space:

$$\bar{\mathcal{S}} = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_6 e_6 \quad (443)$$

the magnitude of this stress vector is

$$|\mathcal{S}| = [\sigma_1^2 + \sigma_2^2 + \sigma_6^2]^{1/2} \quad (444)$$

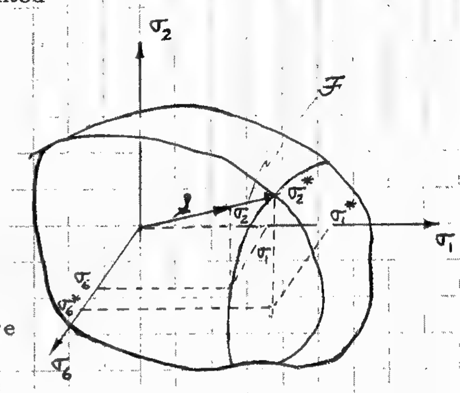


Figure 86 Failure surface.

The potential of this stress vector which can lead to failure depends on its proximity to the failure surface along the <u>same</u> direction as the stress vector. The strength in this direction can be characterized by a strength vector $\mathcal{F}$

$$|\mathcal{F}| = [\sigma_1^{*2} + \sigma_2^{*2} + \sigma_6^{*2}]^{1/2} \quad (445)$$

where σ_1^*, σ_2^*, σ_6^* is the point on the failure surface.

<u>Safety factor</u> can be considered as

$$\frac{\mathcal{S}}{\mathcal{F}} = \frac{1}{\text{Safety factor}} \begin{cases} = 1 & \text{imminent failure} \\ < 1 & \text{no failure} \end{cases} \quad (446)$$

Utility of Failure Criterion

A. For a given state of stress σ_i, interrogate whether failure is imminent

B. For a given state of strain e_i, interrogate whether failure is imminent

C. For a given stress ratio $\frac{\sigma_1}{\sigma_2}$, $\frac{\sigma_1}{\sigma_6}$ etc., what are the failure stresses?

D. For a given state of stress, what is the margin of safety?

Applications for different failure criteria will be illustrated by examples. All examples will use the following mechanical properties of a 0° graphite/epoxy lamina

$$Q_{ij} = \begin{bmatrix} 17.6 & 0.581 & 0 \\ 0.581 & 1.21 & 0 \\ 0 & 0 & 0.760 \end{bmatrix} \times 10^3 \text{ ksi/in/in} \quad (447)$$

$$S_{ij} = \begin{bmatrix} 0.057 & -0.027 & 0 \\ -0.027 & 0.838 & 0 \\ 0 & 0 & 1.32 \end{bmatrix} \times 10^{-3} \text{ in/in/ksi} \tag{448}$$

$$\begin{array}{lll} X_1 = 149 \text{ ksi} & X_2 = 6.3 \text{ ksi} & X_6 = 10.5 \text{ ksi} \\ X_1' = 103 \text{ ksi} & X_2' = -18.2 \text{ ksi} & X_6' = 10.5 \text{ ksi} \end{array} \tag{449}$$

$$\text{Biaxial Strength} \begin{cases} \tilde{\sigma}_1 = 177 \text{ ksi} \\ \tilde{\sigma}_2 = -12.8 \text{ ksi} \end{cases} \tag{450}$$

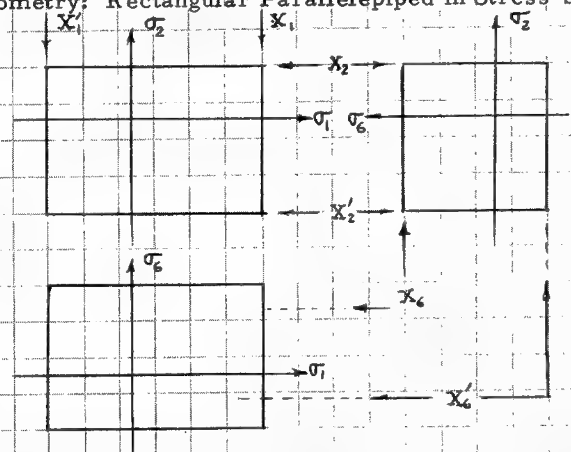
3. FAILURE CRITERIA FOR ONE GIVEN ORIENTATION

NON - Interacting Failure Criteria: biaxial stress does not change uniaxial strength

a. Maximum Stress Criterion

$$
\begin{aligned}
-X_1' \le \sigma_1 &\le X_1 & & \text{a}\\
-X_2' \le \sigma_2 &\le X_2 & & \text{b}\\
-X_6' \le \sigma_6 &\le X_6 & & \text{c}
\end{aligned}
\tag{451}
$$

Geometry: Rectangular Parallelepiped in Stress-Space



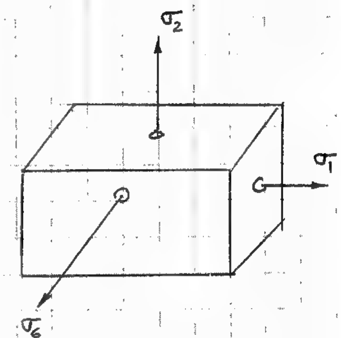


Figure 87 Maximum stress criterion in stress space.

Maximum Stress Criterion in Strain Space

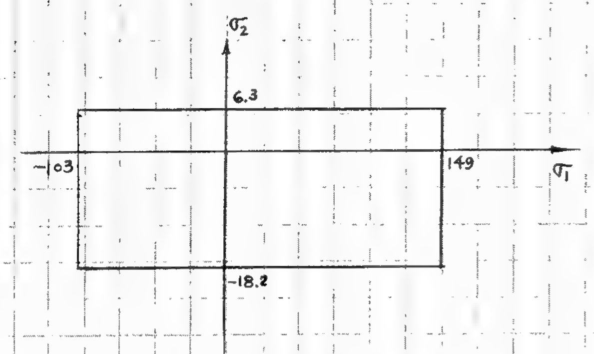
For linearity $\sigma_i = C_{ij} e_j$ Combine with Eq. (451)

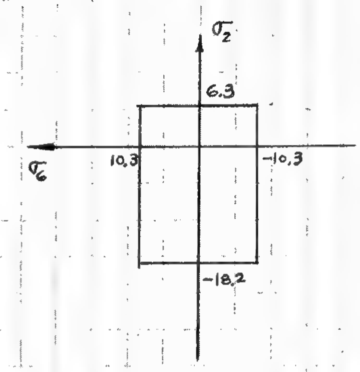
The strain allowables are

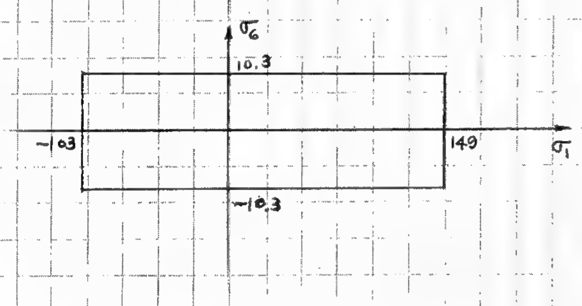
$$
\left.
\begin{aligned}
e_1 &= -\frac{C_{12}}{C_{11}} e_2 - \frac{C_{16}}{C_{11}} e_6 + \frac{X_1}{C_{11}} & & \text{a}\\
e_1 &= -\frac{C_{12}}{C_{11}} e_2 - \frac{C_{16}}{C_{11}} e_6 - \frac{X_1'}{C_{11}} & & \text{b}\\
e_2 &= -\frac{C_{12}}{C_{22}} e_1 - \frac{C_{26}}{C_{22}} e_6 + \frac{X_2}{C_{22}} & & \text{c}\\
e_2 &= -\frac{C_{12}}{C_{22}} e_1 - \frac{C_{26}}{C_{22}} e_6 - \frac{X_2'}{C_{22}} & & \text{d}\\
e_6 &= -\frac{C_{16}}{C_{66}} e_1 - \frac{C_{26}}{C_{22}} e_6 + \frac{X_6}{C_{66}} & & \text{e}\\
e_6 &= -\frac{C_{16}}{C_{66}} e_1 - \frac{C_{26}}{C_{66}} e_6 - \frac{X_6'}{C_{66}} & & \text{f}
\end{aligned}
\right\}
\text{maximum stress criterion in strain space}
\tag{452}
$$

(1) Example

For plane stress condition, plot maximum stress failure criterion in stress space and in strain space using properties given in Section 2.







(2) Example

Given stress σ_i interrogate failure by maximum stress criterion

$\sigma_1 = 100$ ksi

$\sigma_2 = 8$ ksi

$\sigma_6 = 9$ ksi

Substitute into Eq. (451), if any one equation not satisfied, failure occurs

$100 < X_1 = 149$ no failure

$8 > X_2 = 6.3$ failure $\longrightarrow$ Failure for lamina

$9 < X_6 = 10.3$ no failure

(3) Example

Given strain e_i interrogate failure

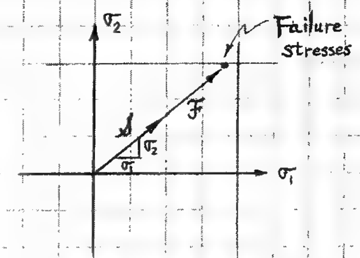
$e_1 = -5.5 \times 10^{-3}$

$e_2 = 6 \times 10^{-3}$

$e_6 = 2 \times 10^{-3}$

(4) Example

For the following state of stress (ratios), using maximum stress criterion, find the failure stresses.



$$\sigma_1 = 70$$

$$\sigma_2 = 3$$

$$\sigma_6 = 5.5$$

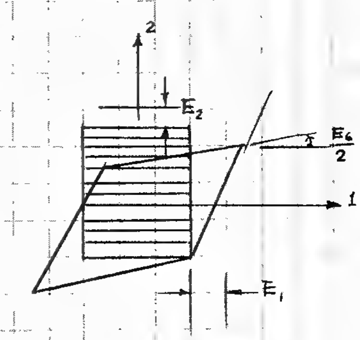
We need to find, along the stress ratios $\frac{\sigma_2}{\sigma_1}, \frac{\sigma_6}{\sigma_1}$, what is the intercept of the failure surface.

Since the maximum stress failure criterion (a rectangular parallelpiped) is a surface bounded 6 sides, the stress vector $\mathcal{S}$ may penetrate any one of the sides. Accounting for the signs, we may interrogate the positive surfaces only.

(5) Example

For the state of stress given in example 4, what is the stress vector, strength vector and the factor of safety predicted by the maximum stress criterion.

b. Maximum Strain Criterion



$$
\begin{aligned}
E_1 &= S_{11}X_1 \geq \epsilon_1 \\
E_1' &= S_{11}X_1' \geq -\epsilon_1 \\
E_2 &= S_{22}X_2 \geq \epsilon_2 \\
E_2' &= S_{22}X_2' \geq -\epsilon_2 \\
E_6 &= S_{66}X_6 \geq \epsilon_6 \\
E_6' &= S_{66}X_6' \geq -\epsilon_6
\end{aligned}
\qquad (453)
$$

E_i Positive Uniaxial Maximum Strain in i component

E_i' Negative Uniaxial Maximum Strain in i component

Geometry: Rectangular Parallelepiped in Strain Space

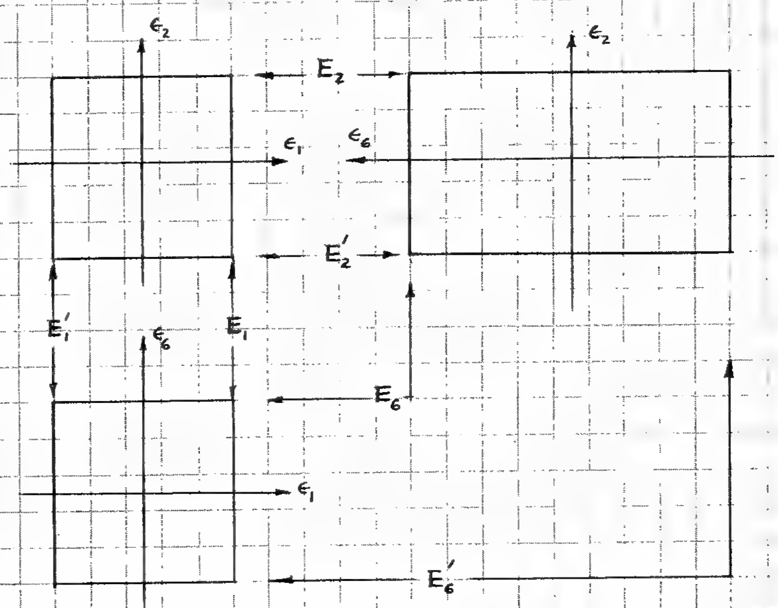


Figure 88 Maximum strain criterion in strain space.

Maximum Strain Criterion in Stress Space

For linearity $e_i = S_{ij}\sigma_j$, combine with Eq. (453).

The allowable stresses are:

$$\left.\begin{aligned}
\sigma_1 &= -\frac{S_{12}}{S_{11}}\sigma_2 - \frac{S_{16}}{S_{11}}\sigma_6 + X_1 && \text{a}\\
\sigma_1 &= -\frac{S_{12}}{S_{11}}\sigma_2 - \frac{S_{16}}{S_{11}}\sigma_6 - X_1' && \text{b}\\
\sigma_2 &= -\frac{S_{12}}{S_{22}}\sigma_1 - \frac{S_{26}}{S_{22}}\sigma_6 + X_2 && \text{c}\\
\sigma_2 &= -\frac{S_{12}}{S_{22}}\sigma_1 - \frac{S_{26}}{S_{22}}\sigma_6 - X_2' && \text{d}\\
\sigma_6 &= -\frac{S_{16}}{S_{66}}\sigma_1 - \frac{S_{26}}{S_{66}}\sigma_2 + X_6 && \text{e}\\
\sigma_6 &= -\frac{S_{16}}{S_{66}}\sigma_1 - \frac{S_{26}}{S_{66}}\sigma_2 - X_6' && \text{f}
\end{aligned}\right\}\text{maximum strain criterion in stress space} \qquad (454)$$

(1) Example

For graphite epoxy lamia, find the maximum strain failure criterion using the given data.

(2) Example

Given stress σ_i, interrogate failure by maximum strain criterion.

$\sigma_1 = 100$ ksi

$\sigma_2 = 8$ ksi

$\sigma_6 = 9$ ksi

(3) Example

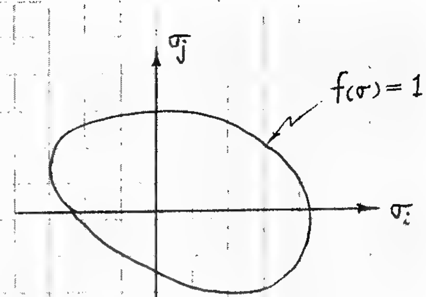
Given strain e_i, interrogate failure by maximum strain criterion.

$e_1 = -5.5 \times 10^{-3}$

$e_2 = 6 \times 10^{-3}$

$e_6 = 2 \times 10^{-3}$

4. INTERACTING FAILURE CRITERION



a. Formulation

$$f(\sigma_i) = F_i\sigma_i + F_{ij}\sigma_i\sigma_j + F_{ijk}\sigma_i\sigma_j\sigma_k + \ldots = 1 \qquad (455)$$

Important features

(1) Each term Scalar product; independent of material coordinate.

(2) Can be expanded to include sufficient (but not excessive) number of terms to describe a given composite

For a 2nd order polynominal i.e.,

$F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1$ expanded for two dimension, i.e., i, j = 1, 2, 6. (456)

The failure tensors F_i, F_{ij} are determined by 9 experiments

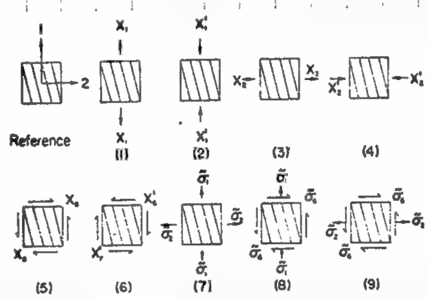
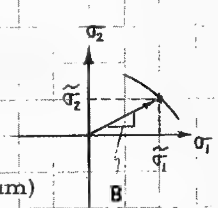
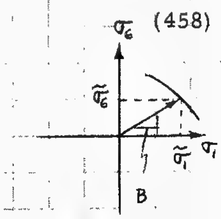


Figure 89 Reference coordinate definition and 9 guiding experiments for measuring failure tensors in a quadratic polynominal failure criterion.

For tests 1 through 6

$$F_i = \left(\frac{1}{X_i}\right) - \left(\frac{1}{X_i'}\right) \quad \text{(no sum)} \qquad (457)$$

$$F_{ii} = \frac{1}{X_i X_i'} \quad \text{(no sum)} \qquad (458)$$

For tests 7 through 9 (or any multiaxial experiments):

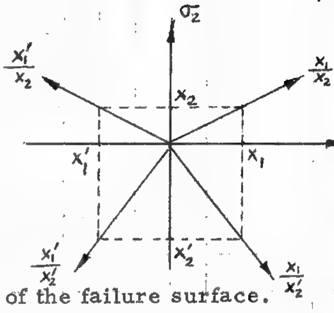
$$(B^2F_{ii} + 2BF_{ij} + F_{jj})\tilde{\sigma}_j^2 + (BF_i + F_j)\tilde{\sigma}_j = 1 \quad \text{(no sum)}$$

$$B = \tilde{\sigma}_i/\tilde{\sigma}_j, \qquad i, j = 1, 2, 6 \qquad (459)$$

The optimal biaxial ratios can be solved from

$$\tilde{\sigma}_j = \frac{-(F_i B + F_j) \pm [(F_i B + F_j)^2 + 4(F_{ii} B^2 + 2F_{ij} B + F_{jj})]^{1/2}}{2(F_{ii} B^2 + 2F_{ij} B + F_{jj})} \tag{460}$$

$$\frac{[2\sigma_j(F_{ii} B + F_{ij}) + F_i]}{[2\sigma_j(F_{ii} B^2 - F_{jj}) - F_j]} = -\frac{[2\sigma_j(F_{ii} B^2 + 2F_{ij} B + F_{jj}) + F_i B + F_j]}{B(BF_i + F_j)} \tag{461}$$



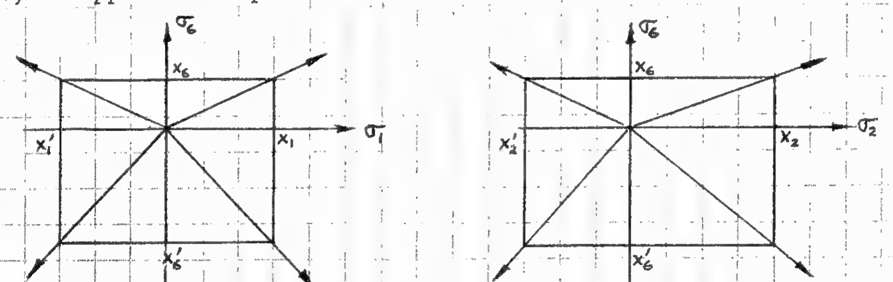
or approximated by

Test 7

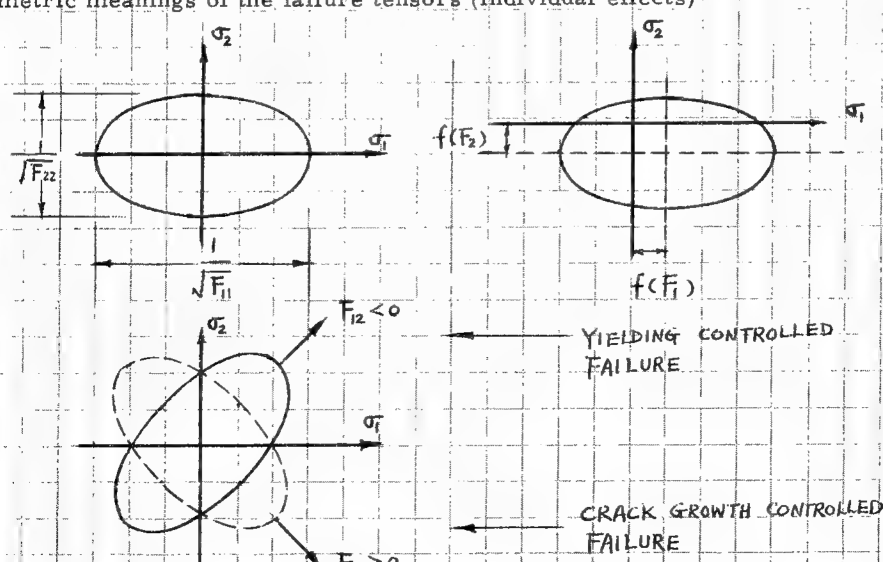
$$B = \frac{X_1}{X_2'} \quad \text{or} \quad \frac{X_1'}{X_2'} \quad \text{or} \quad \frac{X_1'}{X_2} \quad \text{or} \quad \frac{X_1}{X_2} \tag{462}$$

Geometrically these ratios correspond to the four corners of the failure surface.

Similarly the approximate optional ratios for Tests 8, 9 are shown geometrically



Geometric meanings of the failure tensors (individual effects)



Computation using the TENSOR POLYNOMIAL CRITERION (of 2nd order)

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i < 1 \Rightarrow \text{no failure}$$

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1 \Rightarrow \text{failure occurs} \qquad i, j = 1, 2, 6 \tag{463}$$

Plotting the equation $F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1$ in the σ_1, σ_2, σ_6 space, one will obtain a failure envelope in the stress space.

Let:

$$\sigma_1 = R \cos\alpha \cos\beta \qquad a$$

$$\sigma_2 = R \sin\alpha \cos\beta \qquad b \tag{464}$$

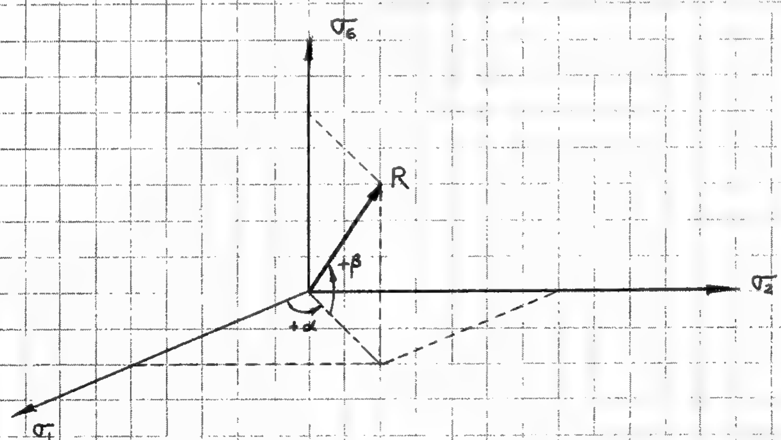
$$\sigma_6 = R \sin\beta \qquad c$$

where:

$$R = [\sigma_1^2 + \sigma_2^2 + \sigma_6^2]^{1/2} \qquad a$$

$$\alpha = \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\right) \qquad b \tag{465}$$

$$\beta = \tan^{-1}\frac{\sigma_6}{\sqrt{\sigma_1^2 + \sigma_2^2}} \qquad c$$



$-180° \leq \alpha \leq 180°$

$-180° \leq \beta \leq 180°$

Figure 90 Polar coordinates.

Thus, we can rewrite $F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1$ as follows:

$$AR^2 + BR = 1 \tag{466}$$

where:

$$A = (F_{11}\cos^2\alpha + F_{12}\sin 2\alpha + F_{22}\sin^2\alpha)\cos^2\beta + F_{66}\sin^2\beta + (F_{16}\cos\alpha + F_{26}\sin\alpha)\sin 2\beta \tag{467}$$

$$B = (F_1\cos\alpha + F_2\sin\alpha)\cos\beta + F_6\sin\beta \tag{468}$$

Solving Eq. (466) for R, taking positive root, we obtain

$$R* = \frac{-B + \sqrt{B^2+4A}}{2A} \tag{469}$$

For an applied stress σ_i, the stress and strength vectors can be computed from Eqs. (465a) and (469), respectively, i.e.:

$\mathcal{S}$ = R (from Eq. (465))

$\mathcal{F}$ = R* (from Eq. (469) where α and β are from Eqs. (465 b, c))

If: $\mathcal{S} < \mathcal{F} \Rightarrow$ no failure

If: $\mathcal{S} \geq \mathcal{F} \Rightarrow$ failure occurs

To compute the $\mathcal{F}$ vector for a given stress ratios, e.g. $\frac{\sigma_2}{\sigma_1}$, $\frac{\sigma_6}{\sigma_1}$, α and β are computed from Eqs. (465b,c) by setting $\sigma_1 = 1$, and $\mathcal{F}$ can be obtained from Eq. (469).

Caution: The signs of the angles α and β should be set according to Figure 90.

b. Computation of the Failure Tensors

Using the strength given in Section IX. 2 and Eqs. (457-458),

$$F_1 = \frac{1}{X_1} - \frac{1}{X_1'} = \frac{1}{149} - \frac{1}{103} = -0.003\ \frac{1}{\text{ksi}}$$

$$F_2 = \frac{1}{X_2} - \frac{1}{X_2'} = \frac{1}{6.3} - \frac{1}{18.3} = 0.105\ \frac{1}{\text{ksi}}$$

$$F_6 = \frac{1}{X_6} - \frac{1}{X_6'} = \frac{1}{10.} - \frac{1}{10.5} = 0$$

$$F_{11} = \frac{1}{X_1X_1'} = \frac{1}{(149)(103)} = 0.065 \times 10^{3}\ \frac{1}{(\text{ksi})^2}$$

$$F_{22} = \frac{1}{X_2X_2'} = \frac{1}{(6.3)(18.3)} = 8.72 \times 10^{-3}\ \frac{1}{(\text{ksi})^2}$$

$$F_{66} = \frac{1}{X_6X_6'} = \frac{1}{(10.5)(10.5)} = 9.07 \times 10^{-3}\ \frac{1}{(\text{ksi})^2}$$

From Eq. (459)

$$B = \frac{\tilde{\sigma}_1}{\tilde{\sigma}_2} = \frac{177}{-12.8} = -13.82$$

$$(B^2F_{11} + 2BF_{12} + F_{22})\tilde{\sigma}_2^{\,2} + (BF_1 + F_2)\tilde{\sigma}_2 = 1$$

$$((-13.82)^2(0.065 \times 10^{-3}) + (2)(-13.82)F_{12} + (8.72 \times 10^{-3}))(-12.8)$$

$$+ (13.82(-0.003) + 0.105) - 12.8 = 1$$

Solve for F_{12} $\qquad F_{12} = 0.20 \times 10^{-3} \frac{1}{(\text{ksi})^2}$

c. Application Examples

(1) Given stress σ_i, interrogate failure condition by tensor polynominal

$\sigma_1 = 50$ ksi
$\sigma_2 = -15$ ksi
$\sigma_6 = 5$ ksi

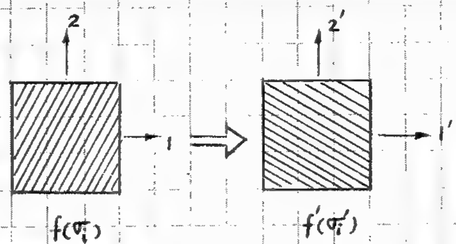
(2) Given strain e_i, interrogate failure condition by tensor polyminal

$$e_1 = -5.5 \times 10^{-3}$$
$$e_2 = 6 \times 10^{-3}$$
$$e_6 = 2 \times 10^{-3}$$

(3) Given stress ratios $\frac{\sigma_2}{\sigma_1} = 0.5$, $\frac{\sigma_6}{\sigma_1} = -0.2$, $\sigma_1 > 0$ compute the strength in this direction of loading

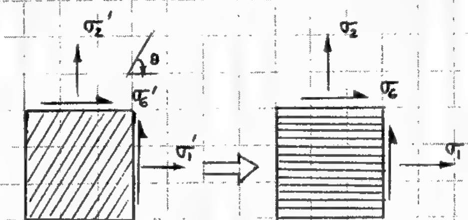
(4) Given $\sigma_1 = 10$, $\sigma_2 = 5$, $\sigma_6 = -2$ ksi, what is the safety factor

5. TRANSFORMATION OF FAILURE CRITERION FROM ONE ORIENTATION TO ANOTHER ORIENTATION.



Method 1

For maximum stress or maximum strain criterion, transform stress (or strain) to reference direction where failure criterion is known.



Apply failure criterion Eq. (451) or (455).

Only one state of stress can be evaluated per operation.

Method 2

For tensor polynominal failure criterion, either use method 1 or transform failure tensors from reference direction to desired direction.

$F_i \Rightarrow F_i'$

$F_{ij} \Rightarrow F_{ij}'$

$$F_i'\sigma_i' + F_{ij}'\sigma_i'\sigma_j' = 1 \qquad (470)$$

Reference direction

Arbitrary direction X_i'

Any state of stress in orientation X_i' can be evaluated

6. LAMINATE STRENGTH

a. First Ply Failure

To assess laminate strength two approaches can be used

(1) Assume laminate as homogeneous, following method outline in Sections 3 and 4, but replace the lamina failure criterion by a laminate failure criterion; i. e., by appropriate numerical value for F_i, F_{ij} and higher order terms when appropriate

(2) Use laminate analysis as discussed in Section V to obtain stress in each layer and interrogate layer by layer using procedure outlined in Section IX. 5. In this case stress vector and strength vector for the k^{th} layer are respectively

$$\mathcal{S}^{(k)} = [(\sigma_1^{(k)})^2 + (\sigma_2^{(k)})^2 + (\sigma_6^{(k)})^2]^{1/2} \tag{471}$$

$$\mathcal{F}^{(k)} = [(\sigma_1{*}^{(k)})^2 + (\sigma_2{*}^{(k)})^2 + (\sigma_6{*}^{(k)})^2]^{1/2} \tag{472}$$

where $\sigma_i{*}^{(k)}$ are the failure stresses of the k^{th} layer from the roots of

$$F_i^{(k)}\sigma_i{*}^{(k)} + F_{ij}^{(k)}\sigma_i{*}^{(k)}\sigma_j{*}^{(k)} = 1 \tag{473}$$

and $F_i^{(k)}$, $F_{ij}^{(k)}$ are the failure tensors for the direction of the k^{th} layer determined by transformation as described in Section III. 4.

Repeating this for all the layers in the laminate, the potential failure layer can be seen graphically in representation such as in Figure 91.

In Figure 91a, the layer $\theta^{(2)}$ is closest to failure and layer $\theta^{(4)}$ has the greatest margin of safety. Further loading would lead to first-ply failure in $\theta^{(2)}$.

Note that margins of safety are in general not equal for all laminates. The layer thickness and/or orientation may be varied to achieve a uniform failure condition as shown in Figure 91b which is the optimal design. The methods for varying these parameters are sizing and mathematical optimization.

The procedure of determining the strength of a laminate configuration, i. e., $\theta^{(k)}$ $h^{(k)}$, undergoing applied loads N_1, N_2, and N_6 is following:

(i) Compute $Q_{ij}^{(k)}$, $F_{ij}^{(k)}$, and $F_i^{(k)}$

(ii) Compute $A_{ij} = \sum Q_{ij}^{(k)} h^{(k)}$

(iii) Compute A_{ij}^{-1}

(iv) Compute $\epsilon_i^o = A_{ij}^{-1} N_j$

(v) Compute $\sigma_i^{(k)} = Q_{ij}^{(k)} e_j^o$

(vi) Compute $\mathcal{S}^{(k)} = R$ (from Eq. (465a))

(vii) Compute $\mathcal{F}^{(k)} = R^*$ (from Eq. (469))

(viii) Comparing $\mathcal{S}^{(k)}$ to $\mathcal{F}^{(k)}$ to determine whether failure has occurred at the $\theta^{(k)}$ layer

1. Example

Given $[45/-45/0/0]_s$; $h^{(0)} + h^{(45)} + h^{(-45)} = 0.200$ inch

$$\left.\begin{aligned} N_1 &= 10. \\ N_2 &= 1. \\ N_6 &= 4. \end{aligned}\right\} \text{(klbf/in)}$$

b. Behavior After First Ply Failure

"Failure" of a ply (any mode) in a laminate "degrades" the laminate but may not produce ultimate failure. Several schemes have been proposed to account for this degradation including:

(a) Total ply discount - assign zero stiffness and strength to the failed ply, all modes.

(b) Mode limited discount - assign zero stiffness and strength to transverse and shear modes if ply failure is in the matrix phase; if fiber phase, discount all modes as (a) above.

(c) Assign residual properties.

(1) Example

Consider a laminate $[90, 0_2, +45, -45]_s$ subject to $N_i = \{3380, 1320, 0\}$ lbs/in.

with $Q_{ij} = \begin{bmatrix} 20 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \times 10^6$ psi and t = 0.005 in. all layers.

Assume a membrane state of stress exists such that

$$\{\varepsilon\}_{x,y} = \{e\}^o_{x,y} = [A]^{-1}\{N\}_{x,y} \quad \text{all layers}$$

Use a maximum strain failure criterion with

$$\begin{cases} E_1 = 0.010, \ -0.010 \\ E_2 = 0.005, \ -0.007 \\ E_6 = 0.015, \ -0.015 \end{cases}$$

LAYER	e_1	e_2	e_{12}	M.S. (1)	M.S. (2)	M.S. (1-2)
0	0.006	0.002	0	0.67	1.5	-
90	0.002	0.006	0	4.0	-0.17	-
+45	0.004	0.004	-0.004	1.5	0.25	2.75
-45	0.004	0.004	0.004	1.5	0.25	2.75

A transverse tension failure is predicted in the 90° ply.

Following degradation scheme (b)

$$Q_{22}^{90} = Q_{12}^{90} = Q_{66}^{90} = 0\ ;\ e_2^{90} = e_6^{90} = 0$$

Recalculate [A] and $\{e\}^k_{1,2}$, evaluate M.S.

Problems — Confirm and complete the above analysis

Repeat using the first scheme

Notes

1. Bending

$$\begin{Bmatrix} e^o \\ k \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{Bmatrix} \bar{N} \\ \bar{M} \end{Bmatrix} \quad \text{(bar denotes inclusion of thermal term)}$$

$$\{e\}^k = \{e^o\} + z^k\{k\}$$

2. Composite laminates are 'statically indeterminate' systems in one or both of two senses:

   (a) structural - a structural member has boundary conditions or geometry such that the constitutive equations are necessary in determining stress resultants at any location.

   (b) lamination - the determination of stresses in any given layer requires a knowledge of the laminate constitutive equations and each layer stiffness.

   ex. homogeneous materials N's, M's $\Rightarrow$ σ's

   layered materials N's, M's $\Rightarrow$ e 's $\Rightarrow$ σ's

3. Each cycle of a stress-strength analysis involving degraded laminate properties will, in general, require a new stress analysis for $\{N, M\}$.

7. SIZING FOR STRENGTH

In Section 6, we outlined the procedure for checking the strengths when the lamination configuration is given. In this section, we discuss the methods for determining the lamination configuration. The lamination configuration consists of the orientation and thickness of each lamina.

Using $\frac{\mathcal{S}}{\mathcal{F}}$ to represent the failure condition of the k^{th} layer lamina, i.e., $\frac{\mathcal{S}^{(k)}}{\mathcal{F}^{(k)}} = 1$ is the state of imminent failure, whereas $\frac{\mathcal{S}^{(k)}}{\mathcal{F}^{(k)}} < 1$ is a no-failure condition. When the lamination configuration is not optimized, as in Figure 91a, each layer would be at different degree from failure. For example layer $\theta^{(2)}$ is close to failure and $\theta^{(4)}$ has the greater margin of safety. It can be seen that the orientation of the individual layers or their thickness or both can be varied such that they will have the same factor of safety. This is illustrated in Figure 91b and is the optimum configuration. This optimum configuration cannot be obtained explicitly for the reasons to be explored; it can, however, be determined by formal optimization and nonlinear programing. We present several direct sizing methods for estimating the optimal configurations.

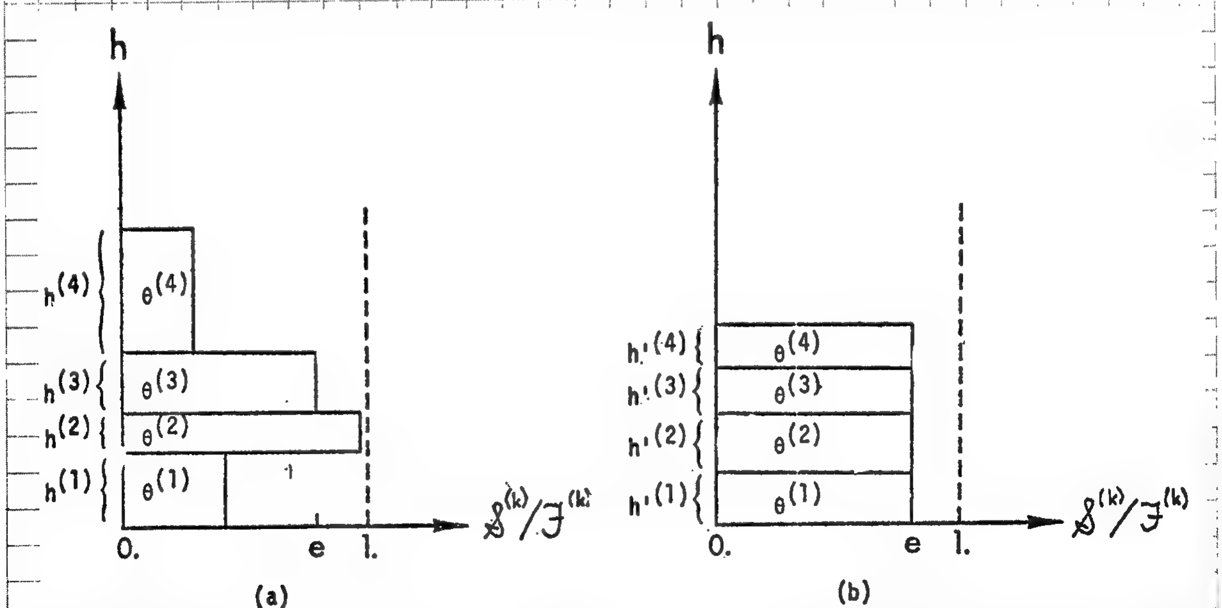
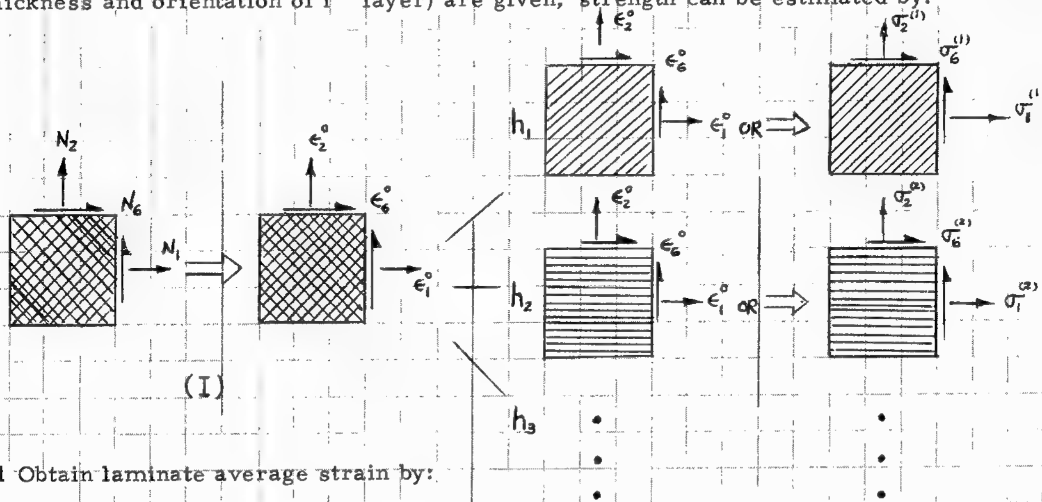


Figure 91 Stress Ratio $\mathcal{S}^{(k)}/\mathcal{F}^{(k)}$ versus the Thickness $h^{(k)}$.

a. Computation for Laminate Strength

In applications where loading configuration N_i and lamination configuration h_i, θ_i (thickness and orientation of i^{th} layer) are given, strength can be estimated by:



I-1 Obtain laminate average strain by:

$Q_{ij}^{(k)} = Q_{ij}^{(k)}(Q_{ij}^{o}, \theta_k)$	(474)	(II)	or	(III)
$A_{ij} = A_{ij}(Q_{ij}^{(k)}, h^{(k)})$	(475)	Apply failure criterion in strain	or	Apply failure criterion in stress
$A_{ij}^{*} = A_{ij}^{*}(A_{ij})$	(476)			
$e_i^{o} = A_{ij}^{*} N_j$	(477)	$g^{(k)}(e_i^{o}) < > 1$		$f(\sigma_i^{k}) < > 1$

- In order to compute A_{ij}^{*}; $h^{(k)}$, $\theta^{(k)}$ must be known
- However, in design where only load carrying capacity N_i is given, $A_{ij}^{*} = A_{ij}(h^{(k)}, \theta^{(k)}, Q_{ij}^{(6)})$ is generally not explicit, the determination of thickness requires either mathematical optimization or direct sizing.

b. Sizing: Strength at Minimum Weight (no pre existing flaws)

Given a generalized loading configuration, determine directly the composite lamination configuration for minimum weight.

Definition of Problem

Given: N_1, N_2, N_6, find $h^{(k)}$ and $\theta^{(k)}$ (thickness and orientation of k^{th} layer)

(1) Method 1: Assume total decoupling of layers and fiber direction carry the load completely.

Limitation for $E_{fiber} \gg E_{matrix}$ (example, fiber reinforced elastomer).

Method 1A: Use 0° and 90° layers exclusively.

Applicability useful for complex but homogeneous state of stress

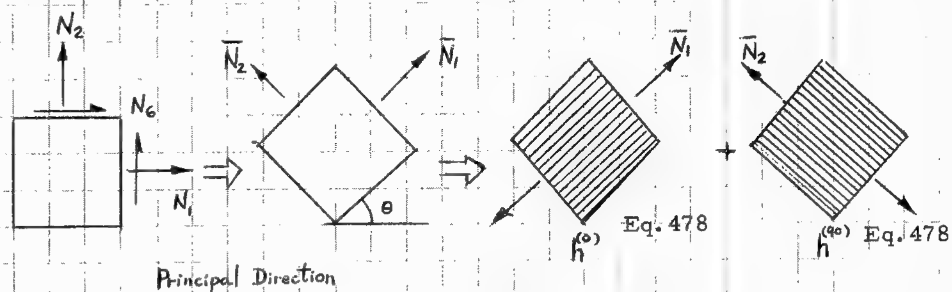


Figure 92 Method 1A.

$$h^{(0)} = \frac{\bar{N}_1}{X_1} \text{ if } \bar{N}_1 > 0 \qquad h^{(90)} = \frac{\bar{N}_2}{X_1} \text{ if } \bar{N}_2 > 0$$

$$= \frac{\bar{N}_1}{X_1'} \text{ if } \bar{N}_1 < 0 \qquad = \frac{\bar{N}_2}{X_1'} \text{ if } \bar{N}_2 < 0 \tag{478}$$

∘ For $e^T_{ult} \geq e^L_{ult}$:

X_1, X_1' are longitudinal tension and compression strength of 0° lamina

∘ For $e^L_{ult} > e^T_{ult}$

$$X_1 = e^T_{ult} Q_{11} \quad , \quad X_1' = -e^T_{ult} Q_{11}$$

Method 1B: Same assumptions and limitations as method 1A.

use 0°, 90°, ± 45 layers

Applicability useful for complex and nonhomogeneous state of stress.

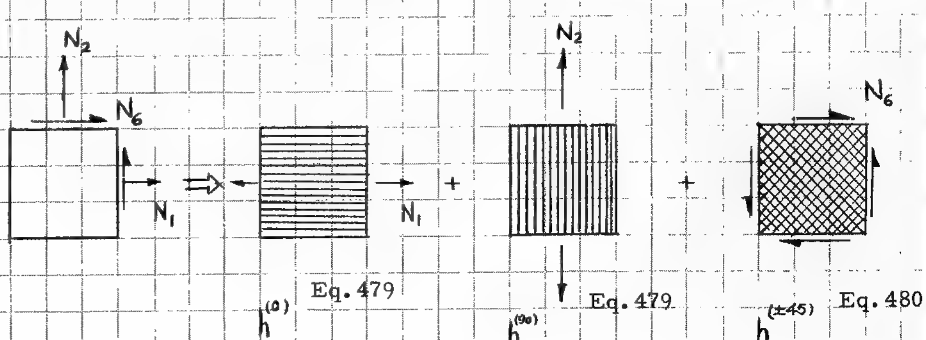


Figure 93 Method 1B.

$$h^{0} = \frac{N_1}{X_1} \text{ if } N_1 > 0 \qquad h^{90} = \frac{N_2}{X_1} \text{ if } N_2 > 0$$

$$= \frac{N_1}{X_1'} \text{ if } N_1 < 0 \qquad = \frac{N_2}{X_1'} \text{ if } N_2 < 0 \tag{479}$$

$$h^{(+45)} = \frac{N_6}{X_1'} \text{ if } N_6 > 0 \qquad h^{(-45)} = \frac{N_6}{X_1} \text{ if } N_6 > 0$$

$$= \frac{N_6}{X_1} \text{ if } N_6 < 0 \qquad = \frac{N_6}{X_1'} \text{ if } N_6 < 0 \tag{480}$$

- For $e^{T}_{ult} \geq e^{T}_{ult}$

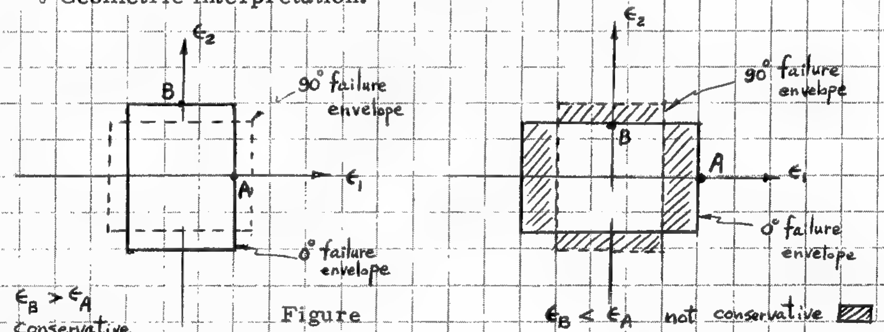
  X_1, X_1' are longitudinal tension and compression strength of 0° lamina

- For $e^{L}_{ult} > e^{T}_{ult}$

  $X_1 = e^{T}_{ult} Q_{11}$, $X_1' = e^{T}_{ult} Q_{11}$

Discussions

- Equivalent to netting analysis, matrix contribution in stiffness and strength are ignored.
- Geometric interpretation.



Figure

- For composite which does not satisfy criterion for being conservative, the actual tensile and compressive strength must be replaced by

  $X_1 = e^{T}_{ult} Q_{11}$

  $X_1' = -e^{T}_{ult} Q_{11}$

c. Explicit Sizing Method (no pre-existing flaws)

(1) Method 2

This method accounts for the interaction between layers by using linear laminated plate analysis. Explicit relationship are presented <u>if</u> lamination orientation are <u>pre-selected</u>.

Definition of Problem:

Given: N_i, e_i*, $\theta^{(k)}$ (stress resultant, failure strain of lamina, lamina orientations)

Find: $h^{(k)}$ (thickness of the k^{th} layer)

General Approach: from classical laminated plate theory, we have

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \qquad (481)$$

n total number of layer

Q_{ij}^{k} stiffness of k^{th} layer

h^{k} thickness of k^{th} layer

$$N_i = A_{ij} e_{ij} \qquad (482)$$

To solve for $h^{(k)}$ explicitly, Eqs. (481-482) are combined as :

[N] = [E] [Q] [H]

where

$$[N] = \begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} \qquad (483)$$

Stress to be carried

$$[E] = \begin{bmatrix} e_1{*} & e_2{*} & 0 & e_6{*} & 0 & 0 \\ 0 & e_1{*} & e_2{*} & 0 & e_6{*} & 0 \\ 0 & 0 & 0 & e_1{*} & e_2{*} & e_6{*} \end{bmatrix} \qquad (484)$$

Failure strain (using max. strain criterion)

$$[Q] = \begin{array}{c} \begin{bmatrix} Q_{11}^{(1)} & Q_{11}^{(2)} & \cdots & Q_{11}^{(n)} \\ Q_{12}^{(1)} & Q_{12}^{(2)} & & Q_{12}^{(n)} \\ Q_{22}^{(1)} & Q_{22}^{(2)} & & Q_{22}^{(n)} \\ Q_{16}^{(1)} & Q_{16}^{(2)} & \cdots & Q_{16}^{(n)} \\ Q_{26}^{(1)} & Q_{26}^{(2)} & & Q_{26}^{(n)} \\ Q_{66}^{(1)} & Q_{66}^{(2)} & \cdots & Q_{66}^{(n)} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ \text{Stiffness of 1st layer} & \text{2nd layer} & & n^{th}\text{ layer} \end{matrix} \end{array} \qquad (485)$$

6 x n (matrix)

(Stiffness of k^{th} layer obtain from Table 20.)

$$[H] = \begin{bmatrix} h^{(1)} \\ h^{(2)} \\ h^{(3)} \\ \vdots \\ h^{(k)} \\ \vdots \\ h^{(n)} \end{bmatrix} \quad \text{n x 1 matrix} \qquad \text{Thickness of each layer} \tag{486}$$

To solve for [H], let [K] = [E] [Q], combining

$$\text{then } [H] = [\,[K]^T [K]\,]^{-1} [K]^T [N] \tag{487}$$

- Eq (487) can be used to determine the necessary thickness for any preselected balanced lamination geometries.
- Explicit solution possible only when non-interactive failure criteria are employed i.e., in Eq. 484 $e_i = e_i{}^*$
- Simplication of Eq. 487 is possible for certain laminate configurations given in

(2) Method 3: Use 0° and 90° layers only for Eq. (487)

Applicability: Homogeneous state of stress, 0° and 90° referred to principal direction.

Compute principal stress $\bar{N}_1$ and $\bar{N}_2$ from given N_i.

$$\text{Eq. (483) becomes} \quad [N] = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \end{bmatrix} \tag{488}$$

$$\text{Eq. (484) becomes} \quad [E] = \begin{bmatrix} e_1{}^* & e_2{}^* & 0 \\ 0 & e_1{}^* & e_2{}^* \end{bmatrix} \tag{488}$$

$$\text{Eq. (485) becomes} \quad [Q] = \begin{bmatrix} Q_{11} & Q_{22} \\ Q_{12} & Q_{12} \\ Q_{22} & Q_{11} \end{bmatrix} \tag{489}$$

$$\text{Eq. (486) becomes} \quad [H] = \begin{bmatrix} h^{(0)} \\ h^{(90)} \end{bmatrix} \tag{490}$$

Eq.(487) becomes $[H] = [K^{-1}] [N]$ (491)

or expanding

$$h^{(0)} = \frac{-(Q_{12} \bar{N}_1 + Q_{22} \bar{N}_1)\, e_1{}^* + (Q_{11} \bar{N}_1 + Q_{12} \bar{N}_2)\, e_2{}^*}{(Q_{11} - Q_{22}) \{Q_{12}(e_1{}^{*2} + e_2{}^{*2}) + (Q_{11} + Q_{22})\, e_1{}^* e_2{}^*\}}$$

$$h^{(90)} = \frac{(Q_{11} \bar{N}_2 - Q_{12} \bar{N}_1)\, e_1{}^* + (Q_{12} \bar{N}_2 - Q_{22} \bar{N}_1)\, e_2{}^*}{(Q_{11} - Q_{22}) \{Q_{12}(e_1{}^{*2} + e_2{}^{*2}) + (Q_{11} + Q_{22})\, e_1{}^* e_2{}^*\}} \quad (492)$$

Inputs to Eq. (491)

Q_{ij} Stiffness of 0 lamina

$\bar{N}_i$ Principal stress from given N_i

$e_1{}^*$, $e_2{}^*$ are determined from Figure 94.

for $\bar{N}_1$, $\bar{N}_2 > 0$ use point A, for $\bar{N}_1 < 0$, $N_2 > 0$ use point B

$\bar{N}_1$, $\bar{N}_2 < 0$ use point C, for $\bar{N}_2 < 0$, $N_1 > 0$ use point D

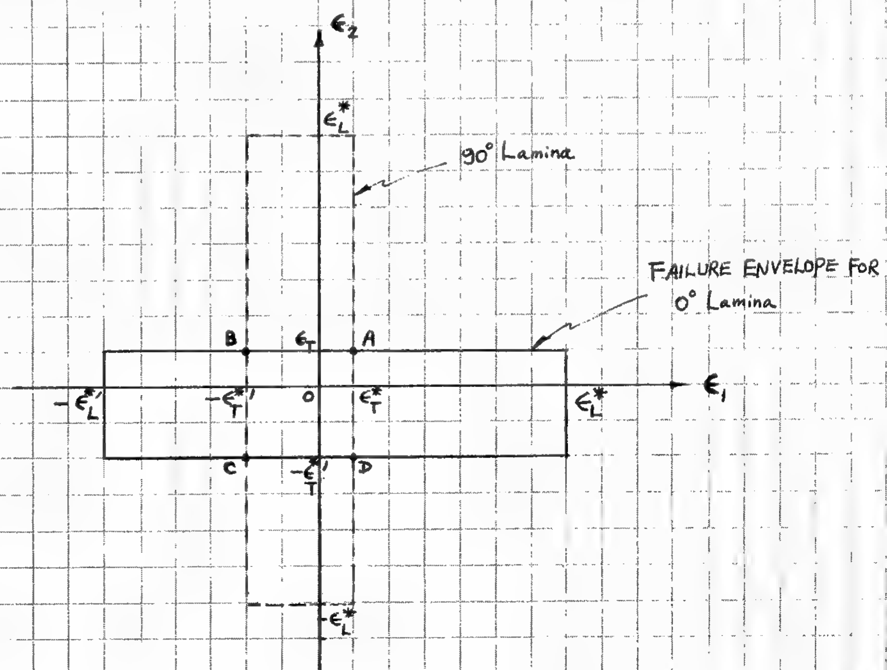


Figure 94 Maximum strain criterion applied to laminate

Example   Scotch-ply 1002   Fiberglass-Epoxy

$$\left.\begin{aligned} Q_{11} &= 0.500 \times 10^{4} \\ Q_{12} &= 0.823 \times 10^{2} \\ Q_{22} &= 0.167 \times 10^{4} \end{aligned}\right\} \text{ksi}$$

For $N_1 = 1000$ lb/in $N_2 = 500$ lb/in

$e_T^* = 1.8 \times 10^{-3}$ $e_L^* = 31 \times 10^{-3}$

$-e_T^* = 12 \times 10^{-3}$ $-e_L^* = 20.2 \times 10^{-3}$

Method 3 Eq.		Method 4 Eq.
$h^{(0)}$	0.111	0.10
$h^{(90)}$	0.056	0.020
Total	0.167	0.120

$e_1^* = 1.8 \times 10^{-3}$

$e_2^* = 1.8 \times 10^{-3}$

(3) Method 5: Use 0/90 and ±45 in pairs.

Applicability: For nonhomogeneous state of stress where a single principal direction does not exist

Eq. (483) unchange
$$[N] = \begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} \tag{493}$$

Eq. (484) becomes
$$[E] = \begin{bmatrix} e_1^* & e_2^* & 0 & 0 \\ 0 & e_1^* & e_2^* & 0 \\ 0 & 0 & 0 & e_6^* \end{bmatrix} \tag{494}$$

Eq. (485) becomes
$$[Q] = \begin{bmatrix} Q_{11}^{(0)} & Q_{22}^{(0)} & Q_{11}^{(45)} \\ Q_{12}^{(0)} & Q_{12}^{(0)} & Q_{12}^{(45)} \\ Q_{22}^{(0)} & Q_{11}^{(0)} & Q_{22}^{(45)} \\ Q_{66}^{(0)} & Q_{66}^{(0)} & Q_{66}^{(45)} \end{bmatrix} \tag{495}$$

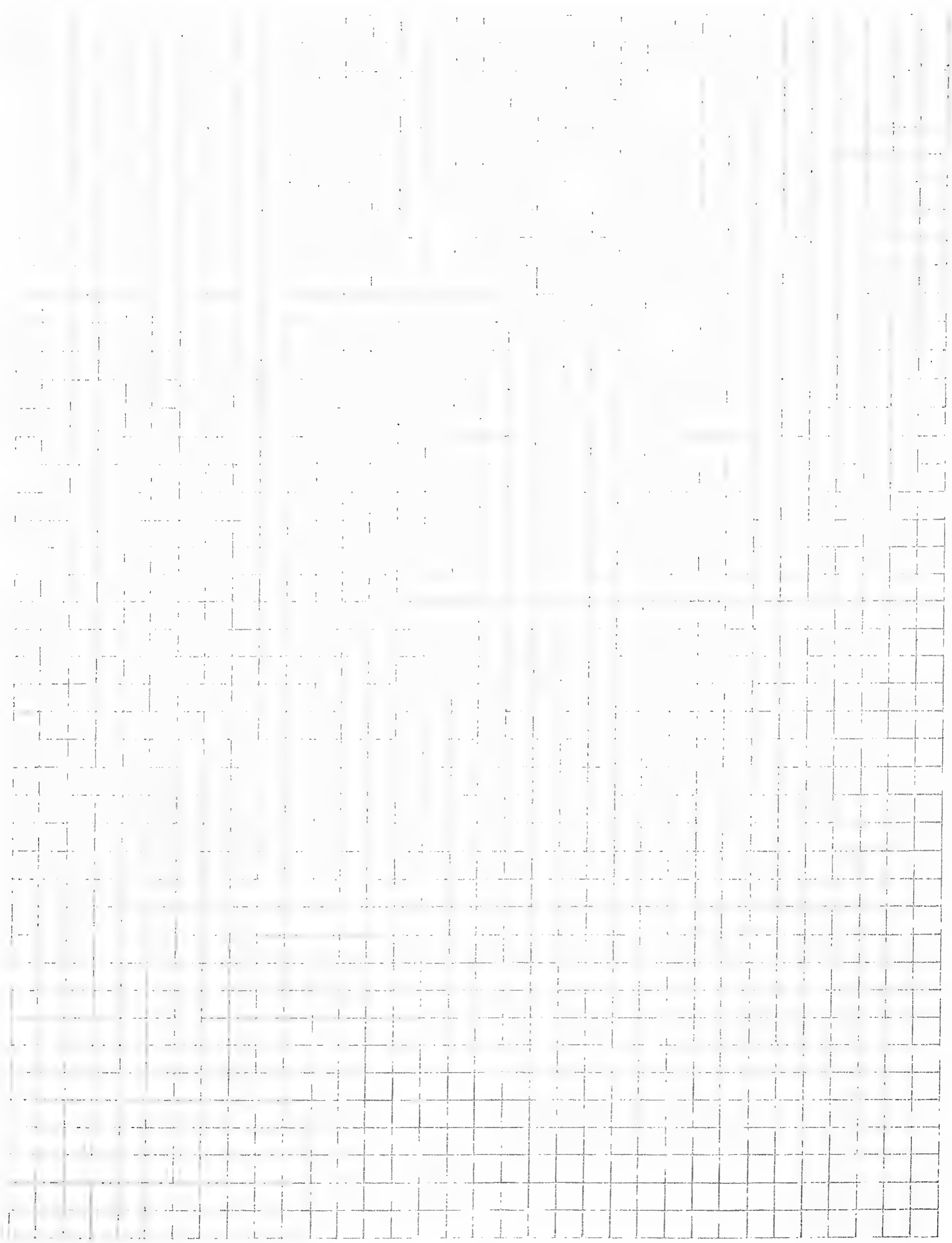
Eq. 486 becomes $[H] = \begin{bmatrix} h^{(0)} \\ h^{(90)} \\ h^{(45)} + h^{(-45)} \end{bmatrix}$ or $\begin{bmatrix} h^{(0)} \\ h^{(90)} \\ 2h^{(45)} \end{bmatrix}$ (496)

Eq. 487 becomes $[H] = [K^{-1}]\,[N]$ (497)

Expanding [K]

$$[K] = \begin{bmatrix} (Q_{11}^{(0)} e_1^* + Q_{12}^{(0)} e_2^*) & (Q_{22}^{(0)} e_1^* + Q_{12}^{(0)} e_2^*) & (Q_{11}^{(45)} e_1^* + Q_{22}^{(45)} e_2^*) \\ (Q_{12}^{(0)} e_1^* + Q_{22}^{(0)} e_2^*) & (Q_{12}^{(0)} e_1^* + Q_{11}^{(0)} e_2^*) & (Q_{12}^{(45)} e_1^* + Q_{22}^{(45)} e_2^*) \\ Q_{66}^{(0)} e_6^* & Q_{66}^{(0)} e_6^* & Q_{66}^{(45)} e_6^* \end{bmatrix} \tag{498}$$

Selection of e_1^*, e_2^*, e_6^* can be obtained from Figure 94 using same rules as presented in Method 5.



SECTION X

FRACTURE TOUGHNESS

1. ELASTIC STRESS ANALYSIS

a. Stress Intensity Factors

$$K_I = \sigma \sqrt{\pi a}\ , \tag{499}$$

$$K_{II} = \tau \sqrt{\pi a}\ . \tag{500}$$

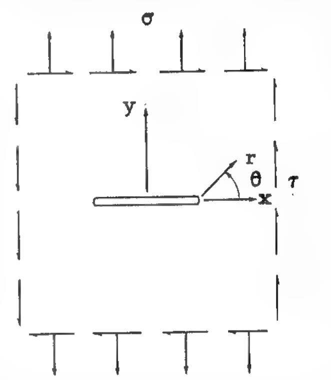


Figure 95 Reference coordinates.

b. Crack Tip Stresses

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left(\frac{\mu_2}{\sqrt{\psi_2}} - \frac{\mu_1}{\sqrt{\psi_1}}\right)\right] + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{\mu_2^2}{\sqrt{\psi_2}} - \frac{\mu_1^2}{\sqrt{\psi_1}}\right)\right] \tag{501}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{\mu_1}{\sqrt{\psi_2}} - \frac{\mu_2}{\sqrt{\psi_1}}\right)\right] + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{1}{\sqrt{\psi_2}} - \frac{1}{\sqrt{\psi_1}}\right)\right] \tag{502}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left(\frac{1}{\sqrt{\psi_1}} - \frac{1}{\sqrt{\psi_2}}\right)\right] + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{\mu_1}{\sqrt{\psi_1}} - \frac{\mu_2}{\sqrt{\psi_2}}\right)\right] \tag{503}$$

$$\psi_1 = \cos\theta + \mu_1 \sin\theta\ , \quad \psi_2 = \cos\theta + \mu_2 \sin\theta \tag{504}$$

μ_1 and μ_2 are the roots of the characteristic equation

$$S_{11}\mu^4 - 2S_{16}\mu^3 + (2S_{12} + S_{66})\mu^2 - 2S_{26}\mu + S_{22} = 0. \tag{505}$$

Note that $\mu_3 = \bar{\mu}_1$, $\mu_4 = \bar{\mu}_2$.

c. Displacements Along $y = 0^+$, $|x| \leq a$

$$u/a = \sigma \operatorname{Re}\left\{\left[S_{11}\mu_1\mu_2 - S_{12}\right]\phi\right\} + \tau \operatorname{Re}\left\{\left[S_{11}(\mu_1+\mu_2) - S_{16}\right]\phi\right\} \tag{506}$$

$$v/a = \sigma \operatorname{Re}\left\{\left[-S_{22}\frac{\mu_1+\mu_2}{\mu_1\mu_2} + S_{26}\right]\phi\right\} + \tau \operatorname{Re}\left\{\left[S_{12} - \frac{S_{22}}{\mu_1\mu_2}\right]\phi\right\} \tag{507}$$

$$\phi = \frac{1}{a}\left[x - i\,(a^2 - x^2)^{1/2}\right] \tag{508}$$

d. Energy Release Rate

$$G = \frac{1}{2}\frac{d}{da}\left(\sigma\int_{-a}^{a} v\,dx + \tau\int_{-a}^{a} u\,dx\right)$$

$$= \frac{1}{2}\left\{K_I^2 \operatorname{Im}\left[-S_{22}\frac{\mu_1+\mu_2}{\mu_1\mu_2} + S_{26}\right] + K_I K_{II} \operatorname{Im}\left[S_{11}\mu_1\mu_2 - \frac{S_{22}}{\mu_1\mu_2}\right]\right. \tag{509}$$

$$\left. + K_{II}^2 \operatorname{Im}\left[S_{11}(\mu_1+\mu_2) - S_{16}\right]\right\} \tag{510}$$

e. Reductions for Orthotropic Materials

$$\mu_1 = \frac{1}{2}\left[n + (n^2 - 4k)^{1/2}\right] i\ , \quad \mu_2 = \frac{1}{2}\left[n - (n^2 - 4k)^{1/2}\right] i \tag{511}$$

$$n = \left[2\left(\sqrt{\frac{S_{22}}{S_{11}}} + \frac{S_{12}}{S_{11}}\right) + \frac{S_{66}}{S_{11}}\right]^{1/2}\ , \quad k = \left(\frac{S_{22}}{S_{11}}\right)^{1/2} \tag{512}$$

$$u = -\sigma(S_{11}k + S_{12})x + \tau S_{11} n (a^2 - x^2)^{1/2} \tag{513}$$

$$v = \sigma S_{22}(n/k)(a^2 - x^2)^{1/2} + \tau(S_{12} + S_{22}/k)x \tag{514}$$

$$G = n\left[(S_{22}/k)K_I^2 + S_{11}K_{II}^2\right] \tag{515}$$

f. Finite Width Correction Factor

Isotropic plates

$$Y = 1 + 0.1282\,(a/w) - 0.2881\,(a/w)^2 + 1.5254\,(a/w)^3 \qquad (516)$$

Orthotropic plates

$$Y\,(\text{ortho.}) \approx Y\,(\text{iso.}) \quad \text{for} \quad L/w > 3.$$

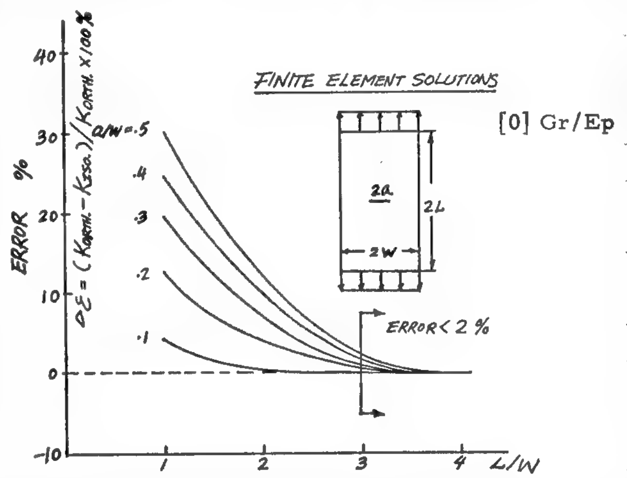


Figure 96 Difference between isotropic and orthotropic finite width correction factors. [21]

g. Particular Cases

$$\theta = 0^\circ$$

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\,k\,, \qquad \sigma_y = \frac{K_I}{\sqrt{2\pi r}}$$

$$\tau_{xy} = 0 \qquad (517)$$

$$\theta = 90^\circ \qquad \mu_1 = \left[n + (n^2 - 4k)^{1/2}\right]/2\,, \quad \mu_2 = \left[n - (n^2 - 4k)^{1/2}\right]/2$$

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\,\frac{k}{\sqrt{2\,\bar{\mu}}}\left(\sqrt{\mu_1} - \sqrt{\mu_2}\right), \qquad \sigma_y = \frac{K_I}{\sqrt{2\pi r}}\,\frac{1}{\sqrt{2\,\bar{\mu}}}\left(\frac{\mu_1}{\sqrt{\mu_2}} - \frac{\mu_2}{\sqrt{\mu_1}}\right) \qquad (518)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\,\frac{k}{\sqrt{2\,\bar{\mu}}}\left(\frac{1}{\sqrt{\mu_1}} - \frac{1}{\sqrt{\mu_2}}\right), \qquad \bar{\mu} = \mu_1 - \mu_2$$

2. FRACTURE TOUGHNESS OF UNIDIRECTIONAL LAMINAE

a. Crack Parallel to Fibers

(1) Mode I and II Loadings

$$\sigma \sqrt{\pi a} = K_{IC} \quad , \qquad \tau \sqrt{\pi a} = K_{IIC} \tag{519}$$

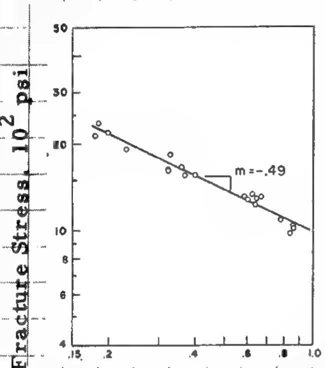


Figure 97 Mode I loading, Scotchply. [22]

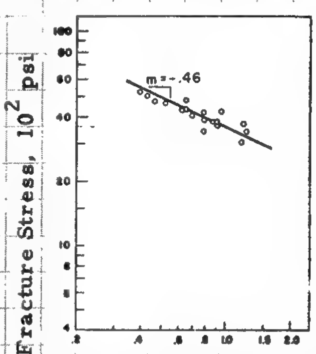


Figure 98 Mode II loading, Scotchply. [22]

(2) Mixed-Mode Loading

Energy criterion

$$G_c = n[(S_{22}/k) K_I^2 + S_{11} K_{II}^2] \tag{520}$$

$$\frac{K_{IIc}}{K_{Ic}} = \left(\frac{S_{22}}{S_{11}}\right)^{1/4} \tag{521}$$

Circumferential stress criterion

$$\sqrt{r}\, \sigma_\theta \Big|_{\theta = o} = \text{const.} \tag{522}$$

Failure strength criterion [23]

$$f(\sigma_i) = F_i \sigma_i + F_{ij} \sigma_i \sigma_j = 1 \quad , \quad \sigma_i = \sigma_i (r_o, \theta) \tag{523}$$

$$\frac{\partial f}{\partial \theta} \Big|_{\theta=\theta_o} = 1 \quad , \quad f(r_o, \theta_o) = 1$$

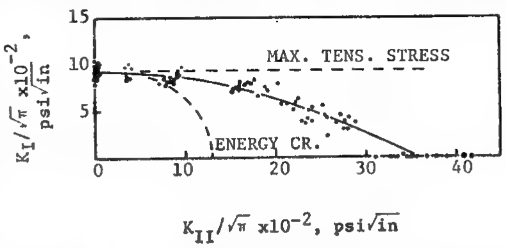
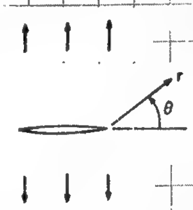


Figure 99 Interaction between K_I and K_{II}, Scotchply (E_L = 34.5 GPa, E_T = 11.5 GPa). [23]

b. Crack Normal to Fibers

(1) Crack Tip Damage

Composites with brittle matrix - longitudinal cracking



$\frac{\sigma_\theta(90)}{\sigma_\theta(0)}$	$\frac{X_T}{X_L}$	$\frac{K_{Ic}(90)}{K_{Ic}(0)}$	$\frac{\tau_{r\theta}(90)}{\sigma_\theta(0)}$	$\frac{S}{X_L}$	$\frac{K_{IIc}(90)}{K_{Ic}(0)}$
0.119	0.051	0.048	0.224	0.066	>0.089

Figure 100 Comparison of crack tip stress ratios with strength and fracture toughness ratios for unidirectional Gr/Ep. Elastic moduli are E_L = 145 GPa, E_T = 11.7 GPa, G_{LT} = 4.48 GPa, ν_{LT} = 0.21. [24]

Composites with ductile matrix - longitudinal plastic deformation

Elastic crack opening displacement (COD) at the center of crack

$$\text{elastic COD} = \frac{\gamma Y a}{E_L}\sigma, \quad \gamma = 2\left[2\left(\sqrt{\frac{E_L}{E_T}} - \nu_{LT}\right) + \frac{E_L}{G_{LT}}\right]^{1/2} \qquad (524)$$

Plastic crack tip opening displacement (CTOD) [25]

$$\text{plastic CTOD} = \frac{\pi}{2}\frac{Y^2 a}{E_L \tau_y}\sigma^2, \quad \tau_y \text{ yield stress of matrix} \qquad (525)$$

Total COD

$$\text{COD} = \text{elastic COD} + \text{plastic CTOD}$$

$$= \frac{Ya\sigma}{E_L}\left(\gamma + \frac{\pi}{2}\frac{Y}{\tau_y}\sigma\right) \qquad (526)$$

For isotropic materials

$$\gamma = 4, \quad 2\tau_y = \sigma_{ys}.$$

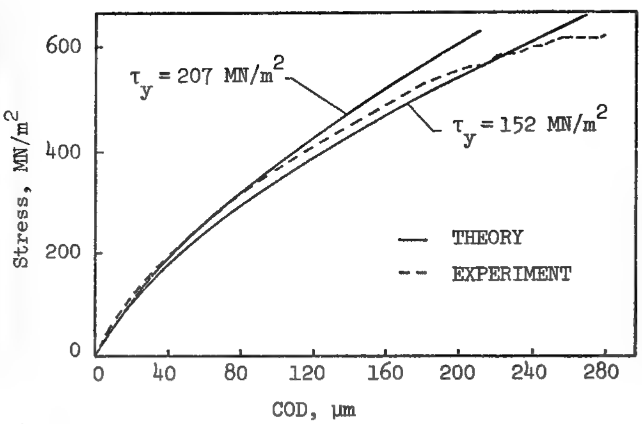


Figure 101 Crack opening displacement in $[0]_{8T}$ B/Aℓ -6061, E_L = 245 GPa, γ = 5.2. [24]

(2) Notch Sensitivity

Longitudinal cracking reduces the notch sensitivity.

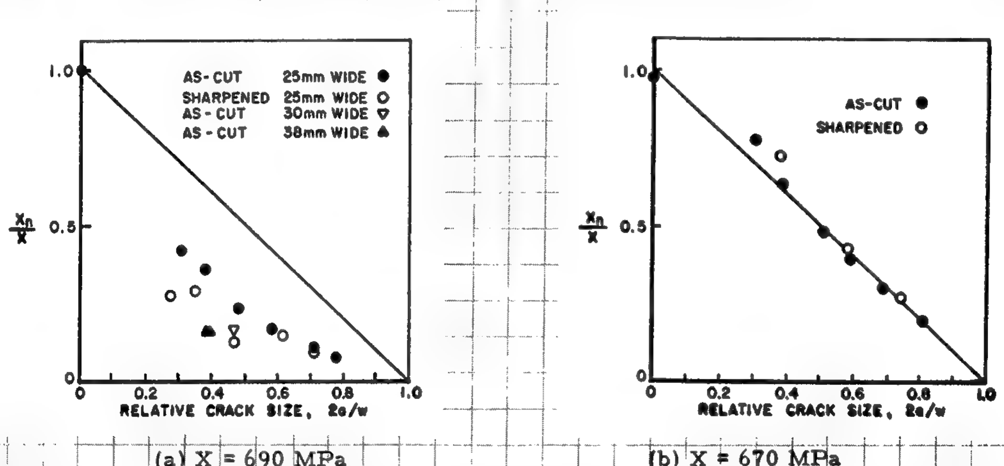


Figure 102 Effect of interfacial bond on notch sensitivity of carbon/epoxy: (a) surface treated; (b) surface untreated. [26]

Longitudinal plastic deformation is much less effective in reducing the notch sensitivity than is the longitudinal cracking.

For B/Al the notched strength is given by

$$\frac{X_n}{X} = \frac{1}{Y}\left(\frac{c_o}{a+c_o}\right)^{1/2} \quad , \quad c_o = 0.908 \text{ mm} . \tag{527}$$

(3) Work of Fracture

W = work accompanying fracture / unit crack extension

Matrix fracture

$$W_1 = W_m(1-v_f) \tag{528}$$

Fiber fracture

$$W_2 = W_f v_f \tag{529}$$

Fiber / matrix debond

$$W_3 = 2W_d\,(L/d_f)\,v_f \tag{530}$$

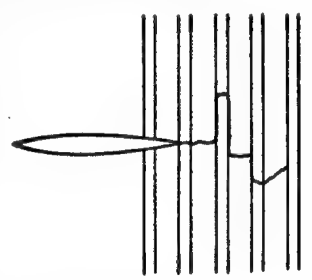


Figure 103 Typical fracture mode in unidirectional composite.

Work after debond [27]

$$W_4 = \frac{4}{3}(\tau_y^2/E_f)\,(L/d_f)^2\,Lv_f \; , \qquad \tau_y\text{: bond strength or friction} \tag{531}$$

or

$$W_4 = \frac{1}{4}(\sigma_f^2/E_f)\,Lv_f \tag{532}$$

Fiber pull-out [28]

$$W_5 = \frac{2}{3}\tau_y d_f (L/d_f)^2 v_f \tag{533}$$

Plastic work in matrix

$$W_6 = \frac{(1-v_f)^2}{v_f}\,\sigma_{my}\,\epsilon_{my}\,d_f \tag{534}$$

L : maximum debond length $\qquad L/d_f = X_f/(4\tau_y)$

Uniform distribution of debond length.

$$W = \sum_{i=1}^{6} W_i \tag{535}$$

3. FRACTURE TOUGHNESS OF MULTIDIRECTIONAL LAMINATES

a. Crack Tip Damage

(1) Nondestructive Examinations (NDE)

Transmitted light - translucent composites (Gl/Ep)

Radiograph with tetrabromoethane - Gr/Ep, B/Ep

Dye penetrant: regular, fluorescent

Ultrasonic C-scan

Reflected light - smooth surface

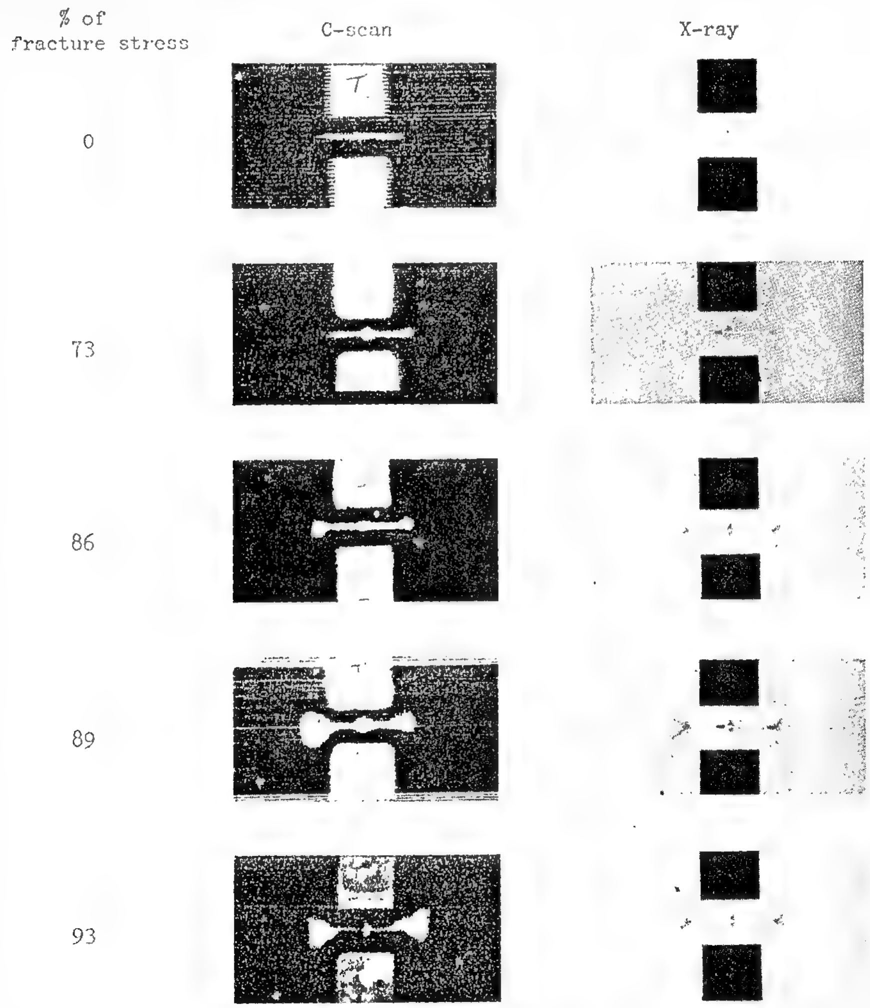


Figure 104 NDE of crack tip damage in $[0/\pm45]_{2s}$ Gr/Ep.

(2) Typical Crack Tip Damage

Subcracks along fiber directions

Delamination between subcracks

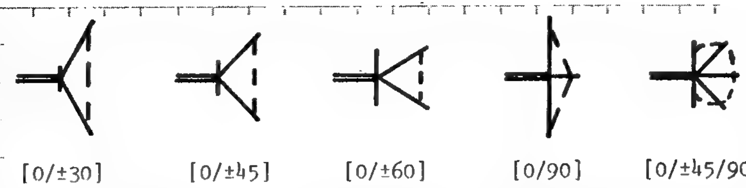


Figure 105 Typical subcracks in [0/±θ] and [0/±45/90] laminates.

(3) Size of Crack Tip Damage Zone (c)

Direct measurement - projected length along the crack plane

Compliance match - compliance based on COD

Plastic zone analogy - analytical

$$K = Y\sigma\sqrt{\pi(a+c)} \quad , \quad c = \frac{1}{2\pi}\left(\frac{K}{\sigma_y}\right)^2 . \tag{536}$$

$$c = \frac{(K'/\sigma_o)^2}{\pi-(K'/\sigma_o)^2/a} \quad , \quad K' = Y\sigma\sqrt{\pi a} \quad , \quad \sigma_y = \sigma_o/\sqrt{2} . \tag{537}$$

TABLE 66 DAMAGE ZONE SIZE (c) IN $[0/\pm 45]_{2s}$ Gr/Ep

σ/σ_n	K'/σ_o	Theory mm	Radiograph mm	C-scan mm
0.73	1.41	0.690	0.762	0.254
0.86	1.66	0.991	1.524	1.778
0.89	1.72	1.075	2.286	2.032
0.93	1.79	1.178	3.302	3.556

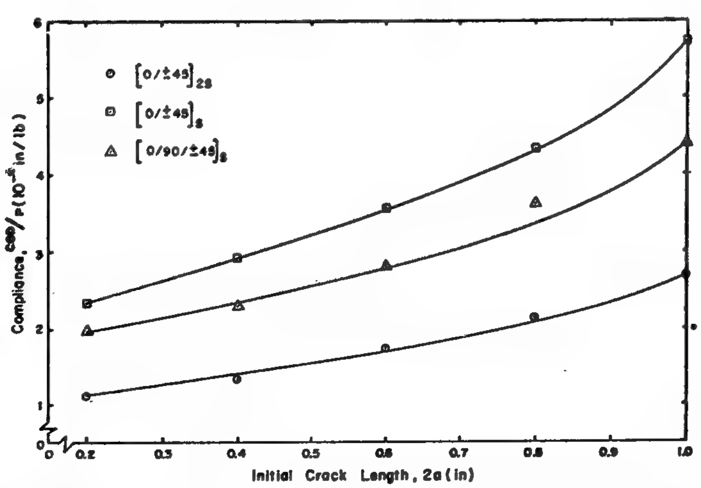
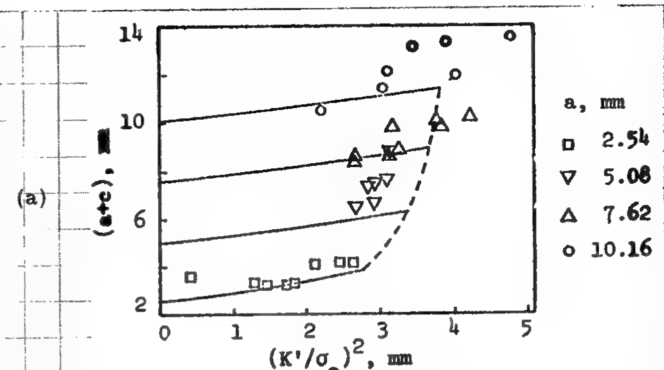


Figure 106 Compliance based on the displacement between two points 3.8 mm above and below crack surface. The material is Gr/Ep. [29]



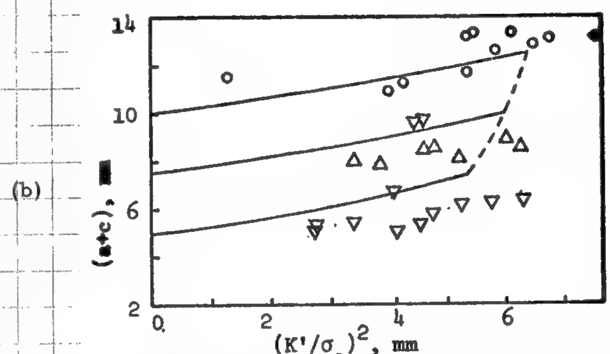


Figure 107 Comparison between analysis and method of compliance match; (a) $[0/\pm 45]_s$ Gr/Ep; (b) $[0/90/\pm 45]_s$ Gr/Gp. [24, 29]

b. Resistance Curves

$$K_R = Y\sigma\sqrt{\pi a_{eff}}\ , \quad a_{eff} = a + c \tag{538}$$

c is determined by the method of compliance match.

c. Prediction of Notched Strength

(1) Final Fracture

$$K_c = YX_n\sqrt{\pi(a+c_o)} \tag{539}$$

c_o can be interpreted as an effective critical damage zone (ECDZ) size.

$$\frac{YX_n}{X} = \left(\frac{c_o}{a+c_o}\right)^{1/2} . \tag{540}$$

This equation also follows from [30]

$$\frac{1}{2c_o}\int_a^{a+2c_o} \sigma_y dx = X . \tag{541}$$

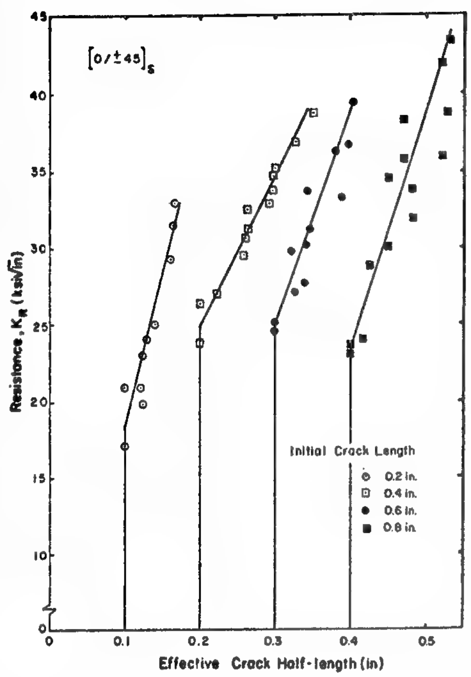


Figure 108 Resistance curves for $[0/\pm 45]_s$ Gr/Ep. Fracture stress is determined from $K = K_R$ and $\partial K/\partial a \geq dK_R/da$. For the composite shown, the terminal point of K_R corresponds to the final fracture. [29]

Define $K_n = YX_n\sqrt{\pi a}$. Then (542)

$$K_n = X\left(\frac{\pi a c_o}{a+c_o}\right)^{1/2} \tag{543}$$

$$K_n = 0 \text{ when } a = 0 . \quad K_n \longrightarrow X\sqrt{\pi c_o} = K_c \text{ as } a \longrightarrow \infty .$$

(2) Damage Initiation

$$K_d = Y\sigma_d\sqrt{\pi a} \ , \quad \sigma_d\text{: stress at damage initiation.} \tag{544}$$

K_d is seen to be fairly independent of a from the resistance curves.

(3) Determination of c_o

c_o is the slope of the following linear equation:

$$a = c_o\left[\left(\frac{X}{YX_n}\right)^2 - 1\right] \tag{545}$$

TABLE 67 ECDZ SIZE AS DETERMINED BY EQ. (545) [24]

Material	Laminate	σ_o MN/m^2	c_o mm	K_c/σ_o $\sqrt{mm}$	N	Range of 2a mm
Gr/Ep T300/5208	$[0/\pm45]_s$ $[0/\pm45]_{2s}$	541	1.372	2.076	27	5-25
	$[0/90/\pm45]_s$	454	2.540	2.825	10	5-25
	$[0/\pm45/90]_s$	494	4.419	3.726	12	2-25
	$[0/90]_{4s}$	637	3.175	3.158	12	2-25
Gr/Ep T300/934	$[\pm45/0/90]_s$	451	3.422	3.279	52	2-15
	$[90/0/\pm45]_s$	499	2.224	2.643	53	2-15
B/Ep	$[0_2/\pm45]_s$ $[0/+45/0/-45]_s$ $[\pm45/0_2]_s$ $[0/+45/0/-45]_{3s}$	802	2.859	2.997	13	1-13
	$[0/\pm45/90]_s$ $[0/\pm45/90]_{4s}$ $[0/90/\pm45]_{4s}$	418	3.176	3.159	9	1-13
Gl/Ep Scotchply	$[0/\pm45/90]_{2s}$	320	1.930	2.462	12	2-25
	$[0/90]_{4s}$	423	1.290	2.013	12	2-25
B/Al 6061-F	$[0]_{8T}$	2004	0.991	1.764	11	1.3-13
BSiC/Ti Ti6-4	$[0]_{6T}$	837	2.206	2.633	12	1.3-13

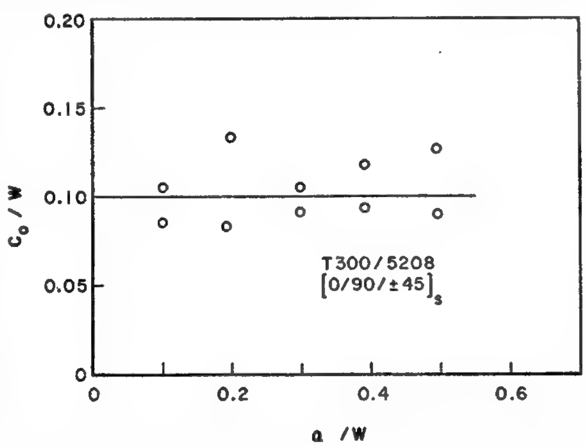


Figure 109 Variation of c_o with crack half-length. [29]

(4) Angle Crack

$$\frac{YX_n}{X} = \left(\frac{c_o}{a\sin\theta + c_o}\right)^{1/2} \tag{546}$$

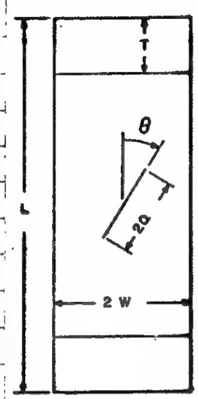


Figure 110 Geometry of specimen with an angle crack.

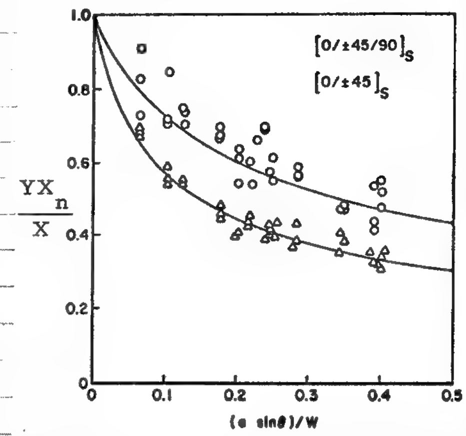


Figure 111 Normalized fracture strength vs. normalized equivalent crack half-length, Gr/Ep.

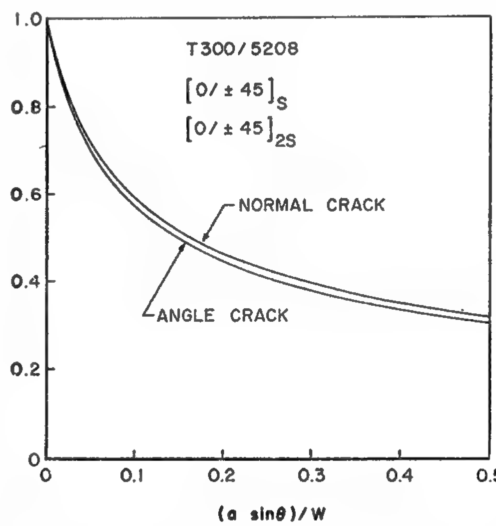


Figure 112 Comparison between normal crack and angle crack, Gr/Ep.

SECTION XI

FATIGUE AND LIFE PREDICTION

1. UNNOTCHED FATIGUE BEHAVIOR

a. Unidirectional Laminae

(1) Longitudinal Fatigue

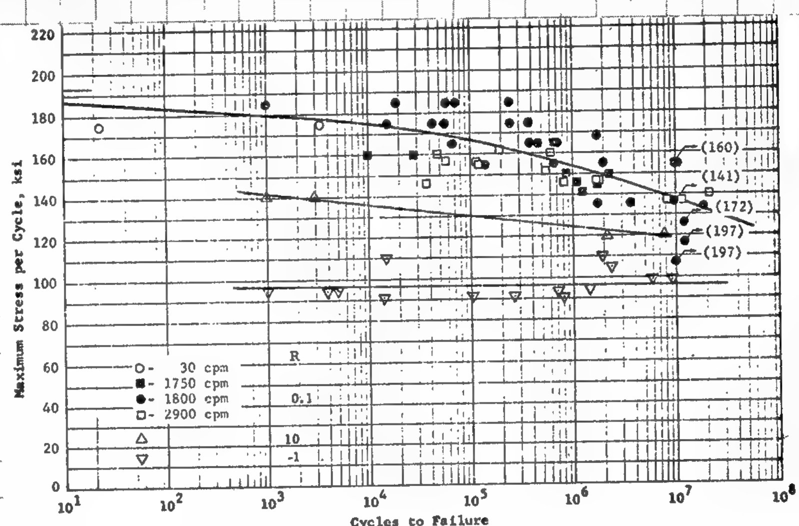


Figure 113 Constant amplitude 0° fatigue of B/Ep at RTD. R=0.1, -1, 10, X=1331 MPa, X'=2206 MPa. [31]

Failure mode

- Extensive longitudinal cracking
- Frequent tab debond
- Failure initiation in the form of longitudinal cracking

Residual strength degradation minimal until immediately before final failure

No change of modulus in B/Ep, Gr/Ep, Gl/Ep.

Flat S-N curve and hence, large scatter.

(2) Transverse Fatigue

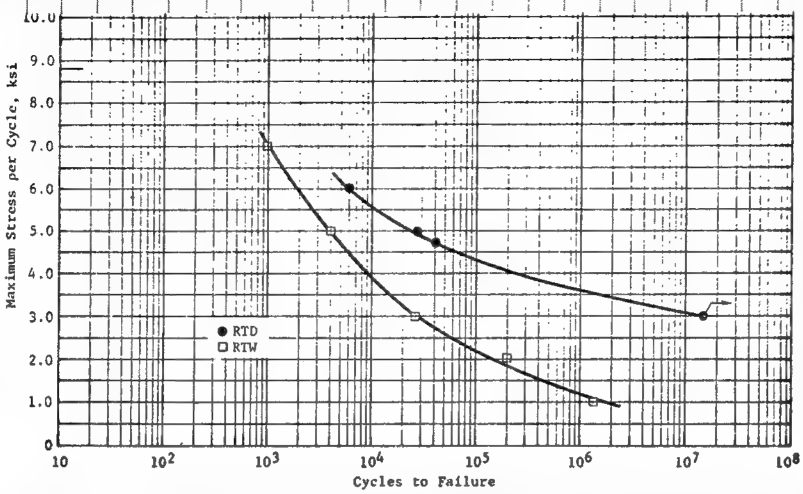


Figure 114 Constant amplitude 90° fatigue of B/Ep at RTD. R=0.1, Y=61 MPa, Y'=276 MPa. [31]

(3) Shear Fatigue

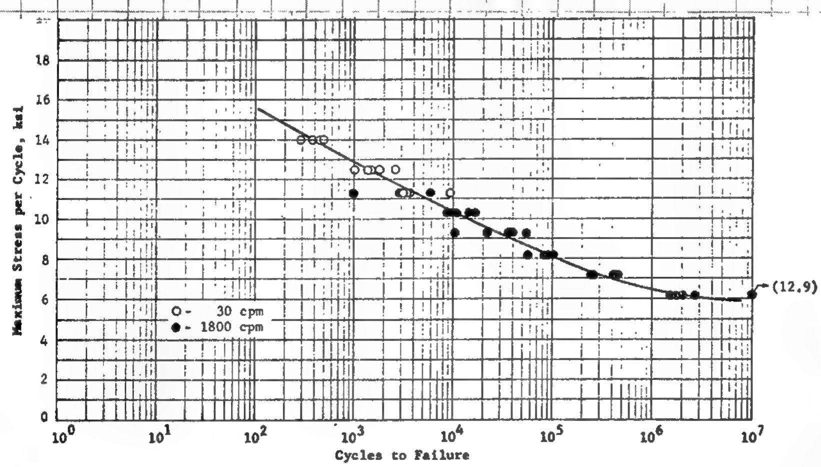


Figure 115 Constant amplitude ±45° fatigue of B/Ep at RTD. R=0.1, X_{45}=133 MPa, $S=X_{45}/2$. [31]

b. Multidirectional Laminates

Fiber controlled laminates

Characteristics of longitudinal fatigue

Matrix controlled laminates

Characteristics of transverse or shear fatigue

(1) S-N Relationships

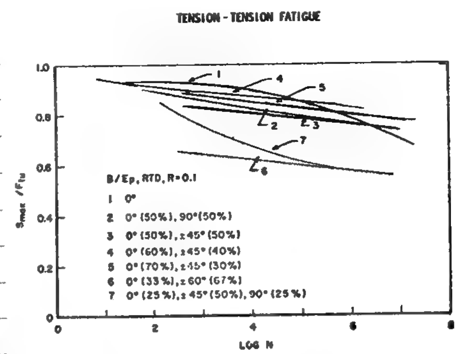


Figure 116 S-N relationships of B/Ep laminates having various fraction of 0° plies, R=0.1. Maximum fatigue stress is normalized with respect to static strength. [31]

Laminate No.	1	2	3	4	5	6	7
Strength, MPa	1331	623	752	779	972	427	338

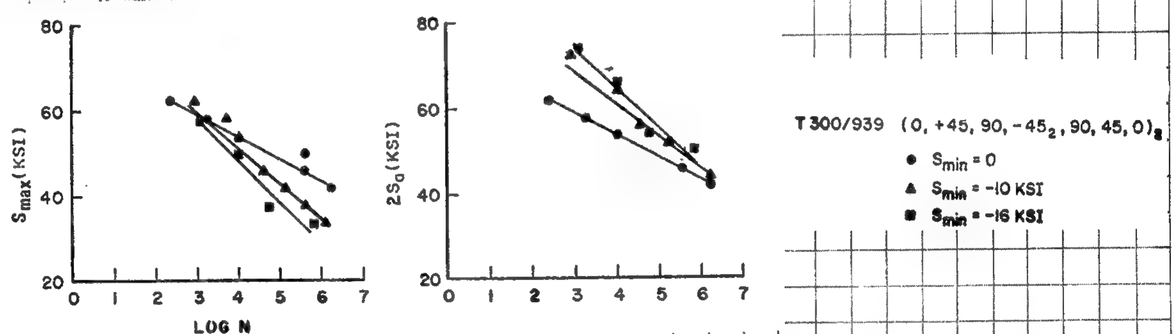


Figure 117 Effect of compression in fatigue of Gr/Ep laminate. [32]

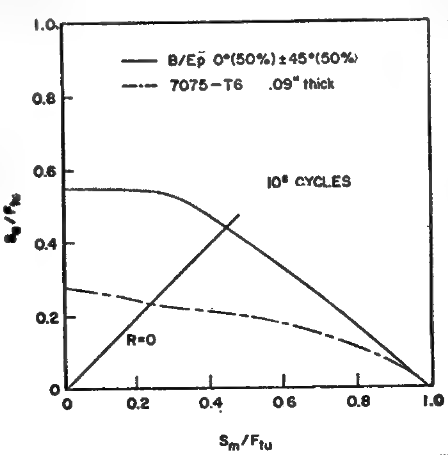


Figure 118 Constant life diagrams for B/Ep laminate and Al-7075-T6.

(2) Failure Mode

Failure sequence in [0/±45/90] laminate

$$\text{Failure of } 90^\circ \text{ ply} \rightarrow \begin{cases} \text{delamination} \\ \text{failure of } \pm 45^\circ \text{ plies} \rightarrow \text{failure of } 0^\circ \text{ ply} \end{cases}$$

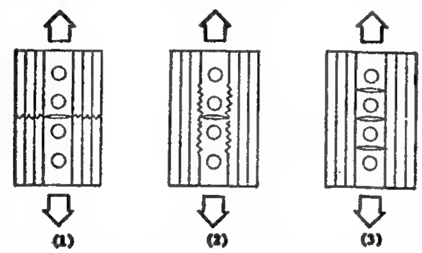


Figure 119 Typical failure mode observed along edge of [0/90] laminate.

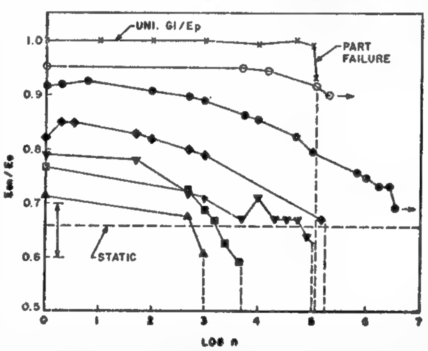


Figure 120 Change of secant modulus in fatigue of $[0/\pm45/90]_S$ Gl/Ep, E_o = 20.3 GPa. [33]

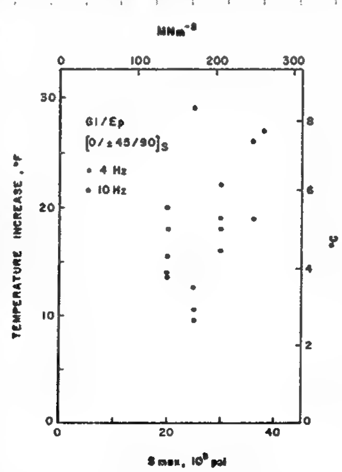


Figure 121 Equilibrium temperature increase in fatigue of Gl/Ep laminate. The increase is proportional to $S^2_{max} f$, where f is the frequency. [33]

2. NOTCHED FATIGUE

a. Failure Mode

Longitudinal cracking at notch tip more extensive than in static tension

More effective relief of stress concentration at crack tip

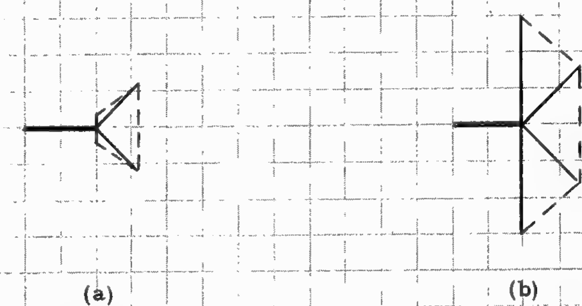


Figure 122 Typical crack tip damage in $[0/\pm45]_S$ Gr/Ep: (a) static tension; (b) fatigue. Dotted lines represent the boundary of delamination.

b. Notch Sensitivity

Residual Strength increases.

Compliance based on notch opening displacement increases.

No penalty for notch.

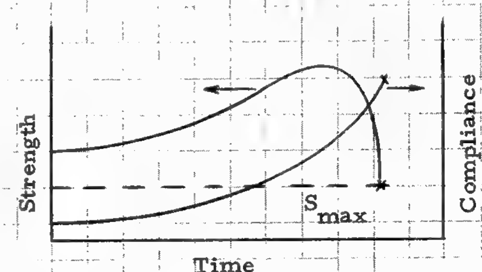


Figure 123 Increase of residual strength and compliance in fatigue of composite laminate (schematic).

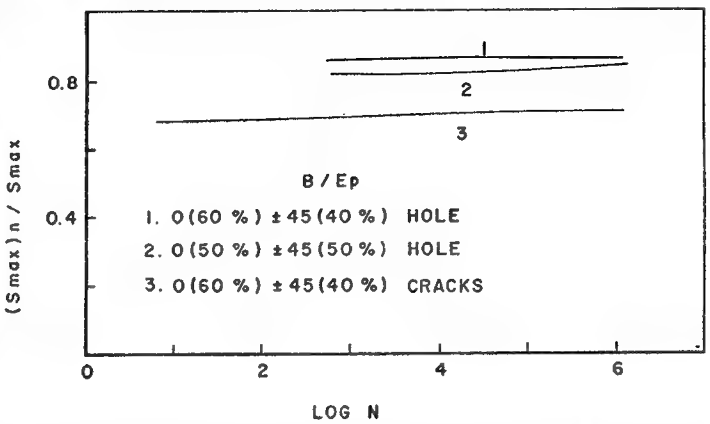


Figure 124 Notched fatigue strength normalized with respect to unnotched fatigue strength. [31]

3. APPROXIMATE PREDICTION FOR FATIGUE LIFE

a. Off-Axis Fatigue Strength

S_L, S_T, S_S : longitudinal, transverse, shear fatigue strength, respectively.

S_θ can be determined by applying any static failure criterion.

For example, the maximum stress criterion of Section II. 2. a. yields

$$S_\theta = \min \left\{ S_L/m^2,\ S_T/n^2,\ S_S/(mn) \right\} \tag{547}$$

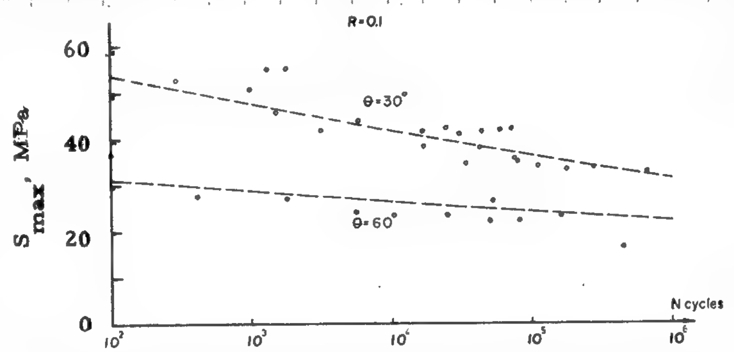


Figure 125 S-N relationships in off-axis fatigue of [o] Gl/Ep. [34]

Assume 30° off-axis fatigue is controlled by shear failure and 60° off-axis fatigue by transverse failure. The data can be fit by

$$S_\theta = -a_\theta \log N + b_\theta \tag{548}$$

θ °	a_θ	b_θ	a_S	b_S	a_T	b_T
	MPa	MPa	MPa	MPa	MPa	MPa
30	7.14	67.97	3.09	29.43	—	—
60	2.29	35.85	—	—	1.72	26.89

For θ=30°, m=cos 30°, n=sin 30°,

$$S_S = mn\, S_{30} = -a_S \log N + b_S \tag{549}$$

$$a_S = mn\, a_{30}, \quad b_S = mn\, b_{30}$$

For $\theta=60°$, $m=\cos 60°$, $n=\sin 60°$

$$S_T = n^2 S_{60} = -a_T \log N + b_T \quad (550)$$
$$a_T = n^2 a_{60}, \quad b_T = n^2 b_{60}$$

Predicted S-N for $\theta=20°$

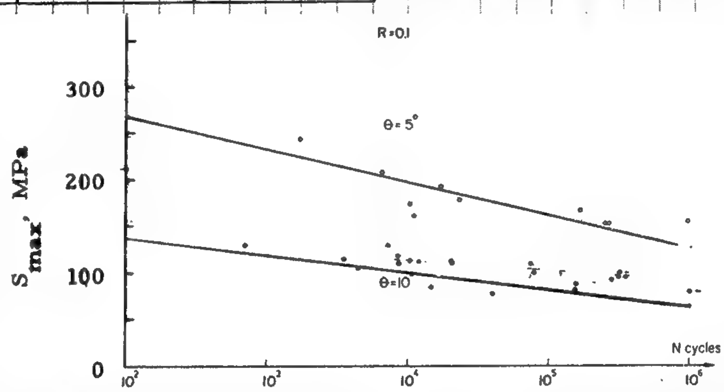
$$S_{20} = -9.614 \log N + 91.57 \text{ MPa} \quad : \text{shear failure} \quad (551)$$

$$S_{20} = -14.70 \log N + 229.87 \text{ MPa} \quad : \text{transverse failure} \quad (552)$$

The first Eq. gives lower fatigue strength and hence represents the S-N relation.

S-N equations based on shear failure

$\theta°$	a_θ, MPa	b_θ, MPa
5	35.59	338.96
10	18.07	172.10
15	12.36	117.72



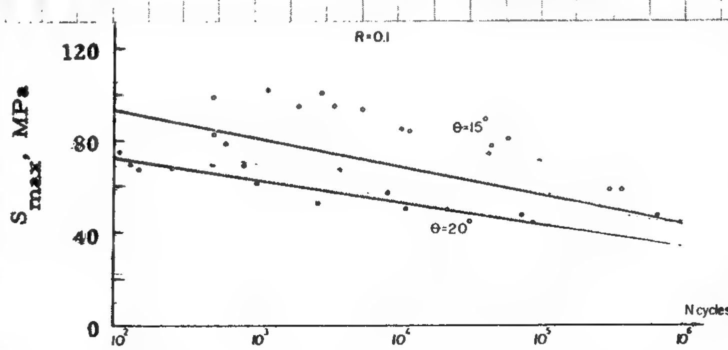


Figure 126 Comparison between data and prediction for off-axis S-N relationship of [0] Gl/Ep. [34]

b. Angle - Ply [$\pm\theta$] Laminates

Assume elastic moduli do not change in fatigue.

Application of the maximum stress criterion to fatigue strengths leads to

$$S_\theta = \min\left\{ S_L/k_L,\ S_T/k_T,\ S_S/k_S \right\} \tag{553}$$

where

$$k_L = \frac{1}{2}\left[1 + \sec 2\theta - \frac{(u_1 + \sec 2\theta)\tan^2 2\theta}{u_2 + \tan^2 2\theta}\right] \tag{554}$$

$$k_T = \frac{1}{2}\left[1 - \sec 2\theta + \frac{(u_1 + \sec 2\theta)\tan^2 2\theta}{u_2 + \tan^2 2\theta}\right] \tag{555}$$

$$k_S = -\frac{1}{2}\ \frac{(u_1 + \sec 2\theta)\tan 2\theta}{u_2 + \tan^2 2\theta} \tag{556}$$

$$u_1 = \frac{1 - E_L/E_T}{1 + 2\nu_{LT} + E_L/E_T} \tag{557}$$

$$u_2 = \frac{E_L/G_{LT}}{1 + 2\nu_{LT} + E_L/E_T} \tag{558}$$

$$S_\theta = -a_\theta \log N + b_\theta \tag{559}$$

$$a_\theta = a_i/k_i \qquad i = L,\ T,\ \text{or } S \tag{560}$$

$$b_\theta = b_i/k_i \tag{561}$$

For Gl/Ep

E_L	E_T	G_{LT}	ν_{LT}	u_1	u_2
GPa	GPa	GPa			
54.7	17.75	6.01	0.285	-0.4475	1.957

$\theta°$	k_L	k_T	k_S	a_θ	b_θ	Type of failure
30			0.271			S
45			0.5	[illegible]	[illegible]	S
60			0.428	[illegible]	[illegible]	T

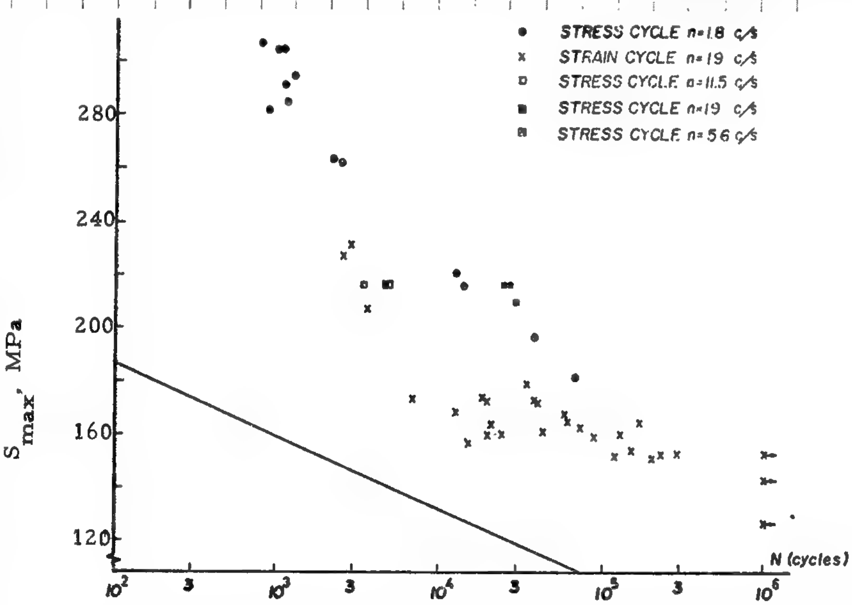


Figure 127 Experimental and predicted S-N curves of [±30] laminate. [35]

c. Fiber-Controlled Laminates

Fatigue strength of laminate (tension-tension)

$$= S_L \times (\text{fraction of } 0^\circ \text{ plies}) \tag{562}$$

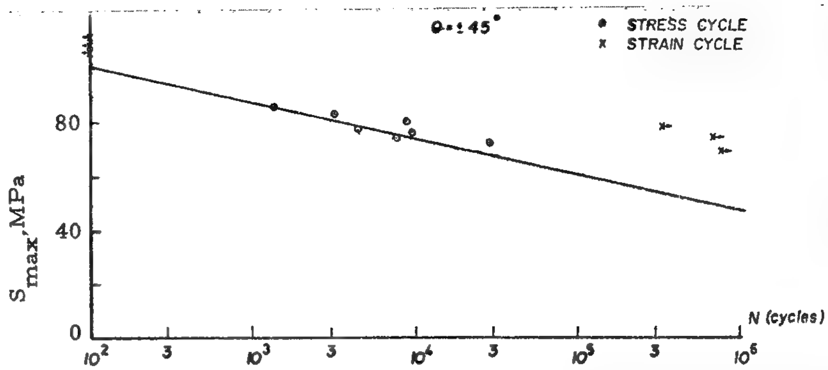


Figure 128 Experimental and predicted S-N curves of [±45] laminate. [35]

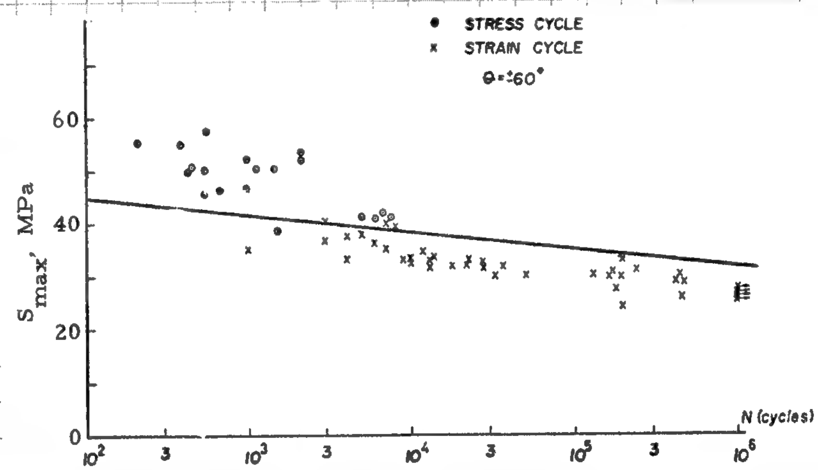


Figure 129 Experimental and predicted S-N curves of [±60] laminate. [35]

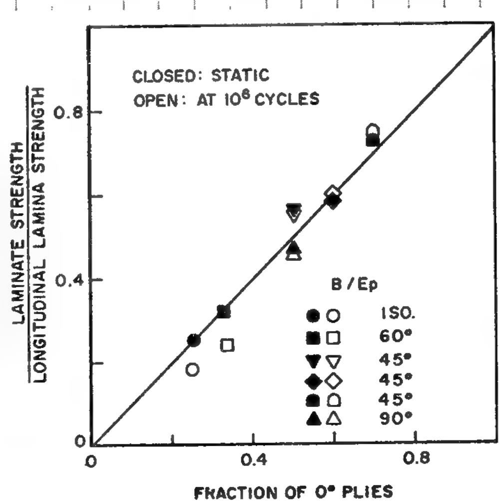


Figure 130 Fatigue strength of laminate is approximately equal to the longitudinal fatigue strength multiplied by the fraction of 0^{o} plies. [31]

4. LIFE PREDICTION AND ANALYSIS OF SCATTER

a. Application of Failure Potential

(1) General Formulation

Probability of surviving time t under a loading history $s(t)$:

$$R(t) = \exp\left[-\psi(\tau)\right] \tag{563}$$

$$\tau = \frac{1}{\hat{t}} \int_0^t K(s(\xi))\, d\xi \text{ , } \hat{t} \text{ has the dimension of } t. \tag{564}$$

Failure potential

$$\psi(\tau) = \tau^{\alpha} \tag{565}$$

Breakdown rules [36]

$$\text{Power law: } K(s) = (s/C_1)^{\beta} \tag{566}$$

$$\text{Exponential law: } K(s) = \frac{1}{C_2} \exp(s/C_3) \tag{567}$$

(2) Stress Rupture

$s(\xi) = s$ const.

Power law breakdown rule

$$\tau = \frac{1}{\hat{t}} \int_0^t (s/C_1)^{\beta} d\xi = (s/C_1)^{\beta}\, t/\hat{t} \tag{568}$$

$$\therefore R(t) = \exp\left[-(s/C_1)^{\beta\alpha} (t/\hat{t})^{\alpha}\right] = \exp\left[-(t/t_o)^{\alpha}\right] \tag{569}$$

$$(t_o/\hat{t})\,(s/C_1)^{\beta} = 1 \tag{570}$$

or $\log(t_o/\hat{t}) + \beta \log s = \beta \log C_1$: Power law (log stress - log time) representation of S-N relationship

Exponential breakdown rule

$$\tau = \frac{1}{\hat{t}} \int_0^t \frac{1}{C_2} \exp(s/C_3) d\xi = \frac{1}{C_2} \exp(s/C_3)\, t/\hat{t} \tag{571}$$

$$\therefore R(t) = \exp\left\{-\left[\frac{1}{C_2} \exp(s/C_3)\right]^{\alpha} (t/\hat{t})^{\alpha}\right\} = \exp\left[-(t/t_o)^{\alpha}\right] \tag{572}$$

$(t_o/\hat{t}) \exp(s/C_3) = C_2$: Exponential (log time - log stress) representation of S-N relationship

$$\log(t_o/\hat{t}) + \left(\frac{\log e}{C_3}\right) s = \log C_2 \tag{573}$$

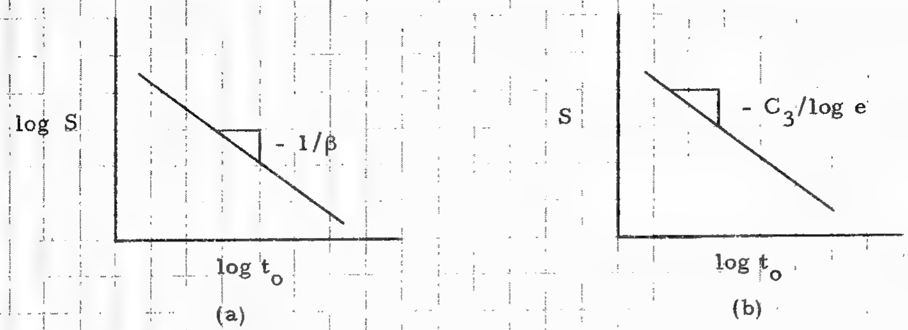


Figure 131 Typical representations of S-N relationship.

(3) Tension With Constant Loading Rate

$$s(\xi) = L\xi \tag{574}$$

Power law breakdown rule

$$\tau = \frac{1}{\hat{t}}\int_0^t \left(\frac{L}{C_1}\right)^\beta \xi^\beta \, d\xi = \frac{(L/C_1)^\beta}{\beta+1} \, t^{\beta+1}/\hat{t} \tag{575}$$

$$R(t) = \exp\left[-\left(\frac{(L/C_1)^\beta}{\hat{t}\,(\beta+1)}\right)^\alpha t^{\alpha(\beta+1)}\right] = \exp\left[-(t/t_o)^{\alpha(\beta+1)}\right] \tag{576}$$

$$t_o = \left[\frac{(\beta+1)\,\hat{t}}{(L/C_1)^\beta}\right]^{1/(\beta+1)} \tag{577}$$

Let $x = Lt$ (578)

$$R(x) = \exp\left[-\left(\frac{x}{x_o}\right)^{\alpha(\beta+1)}\right] \tag{579}$$

$$x_o = Lt_o = \left[\hat{t}\,L\,C_1{}^\beta(\beta+1)\right]^{1/(\beta+1)} \tag{580}$$

Exponential breakdown rule

$$\tau = \frac{1}{\hat{t}}\int_0^t \frac{1}{C_2} \exp\,(L\xi/C_3)d\xi = \frac{C_3}{\hat{t}\,LC_2}\left[\exp(Lt/C_3) - 1\right] \tag{581}$$

$$R(t) = \exp\left\{-\left(\frac{C_3}{\hat{t}LC_2}\right)^\alpha \left[\exp(Lt/C_3) - 1\right]^\alpha\right\} \tag{582}$$

$$R(x) = \exp\left\{-\left(\frac{C_3}{\hat{t}LC_2}\right)^\alpha \left[\exp(x/C_3) - 1\right]^\alpha\right\} \tag{583}$$

In the failure range $\exp(x/C_3) << 1$.

$$\therefore R(x) = \exp\left\{-\left(\frac{C_3}{\hat{t}LC_2}\right)^{\alpha} \exp(\alpha x/C_3)\right\} \qquad (584)$$

$$\bar{x} = \frac{C_3}{\alpha}\left[\alpha \ln(\hat{t}LC_2/C_3) - \gamma\right] \qquad \gamma: \text{ Euler constant } (=0.5772) \qquad (585)$$

TABLE 68 PARAMETERS FOR STRESS-RUPTURE DATA [37]

Strand Type	v_f %	Power Law			Exponential Law	
		β	C_1 MN/m^2	α	C_3 MN/m^2	C_2
Kevlar 49/Ep	71.5	42	2239	0.87	50.5	1.86×10^{19}
Gr/Ep	62	78	922	0.30	10.8	8.32×10^{36}
S-Gl/Ep	65	30	2109	0.75	62.9	1.91×10^{14}
Be/Ep	66	26	733	3.75	25.1	3.06×10^{12}

TABLE 69 AVERAGE TENSILE STRENGTH AND COEFFICIENT OF VARIATION [37]

Strand Type	L $MN/m^2/h$	Average Strength			Coefficient of Variation	
		Power	Exponential	Experiment	Power	Experiment
Kevlar 49/Ep	2.96×10^5		2645	2940		0.025
Gr/Ep	6.32×10^5			915		0.05
S-Gl/Ep	1.34×10^5			2560		0.036
Be/Ep	5.46×10^5			769		0.005

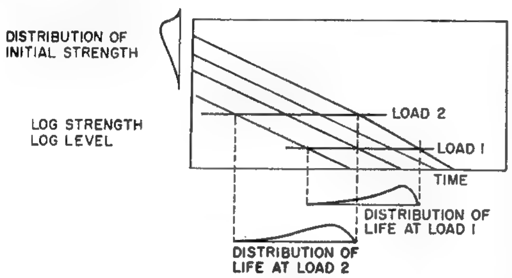
b. Concept of Similarity and Nonsimilarity

(1) Definitions [38]

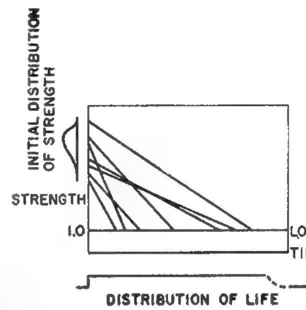
Objects that follow the same law of behavior are similar.

Objects that follow different laws of behavior are nonsimilar.

Two objects are identical if there is an exact duplication of the atoms of the first one in the second.

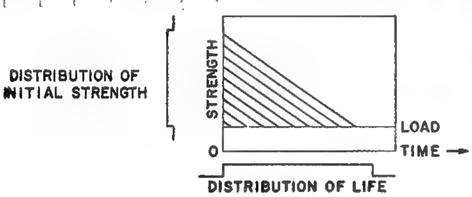


(a)

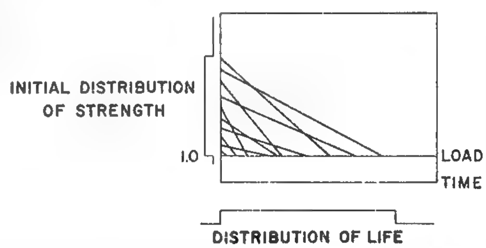


(b)

Figure 132 Life distribution of similar objects: (a) similar deterioration; (b) nonsimilar deterioration.



(a)



(b)

Figure 133 Life distribution of nonsimilar objects: (a) similar deterioration; (b) nonsimilar deterioration.

(2) Relationship Between Static Strength and Life of Similar Objects With Similar Deterioration

(a) Pairing of static strength and life

Static Strength Data

$s \leq x_1 \leq x_2 \leq - - - \leq x_n$, s : fatigue stress

Life Data

$t_1 \leq t_2 \leq - - - \leq t_n$

Strength and life of jth object : (x_j, t_j)

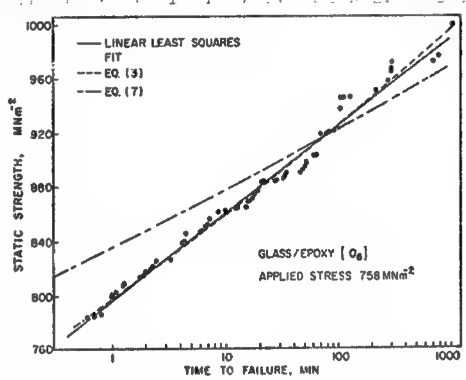


Figure 134 Plot of (t_{ji}, x_j) for [0] Gl/Ep subjected to constant stress. [39]

(b) Similarizing operation (proof testing)

Proof load to x_i. Then the minimum guaranteed life corresponding to x_i is t_i

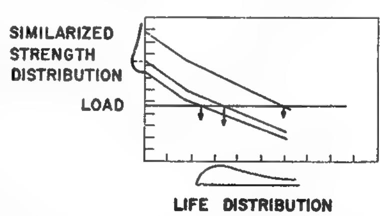


Figure 135 Effect of similarizing operation on life distribution.

(c) Mathematical formulation

Determine strength and life reliability functions

$R_s(x)$ and $P_\ell(t)$

Relationship between x and t is obtained from

$$\bar{R}_s(x|s) = R_\ell(t) \tag{586}$$

where $\bar{R}_s(x|s) = R_s(x) / R_s(s)$ (587)

In particular, if

$$R_s(x) = \exp\left[-(x/x_o)^{\alpha_s}\right] \tag{588}$$

$$R_\ell(t) = \exp\left[-(t/t_o)^{\alpha_\ell}\right] \tag{589}$$

then

$$t = t_o\left[(x/x_o)^{\alpha_s} - (s/x_o)^{\alpha_s}\right]^{1/\alpha_\ell} \tag{590}$$

In terms of residual strength at t

$$\bar{R}_s(x|s) = R_r(x_r) \tag{591}$$

where

$$R_r(s) = R_\ell(t) \tag{592}$$

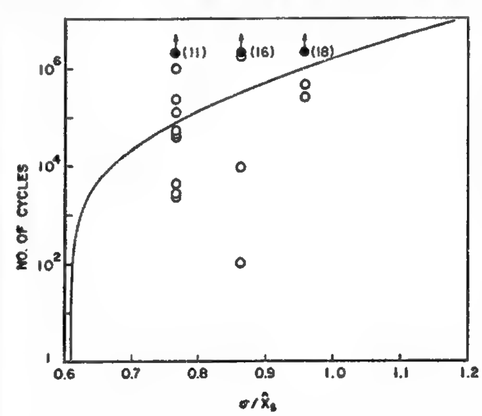


Figure 136 Relation between static strength and fatigue life of [o] Gr/Ep: $\alpha_s = 7.77$, $x_o = 1.44$ GPa, $\alpha_\ell = 0.74$, $t_o = 1.6\text{x}10^6$ cycles, $S_{max}/x_o = 0.61$. Results of proof testing are also shown. [40]

c. Strength Degradation Model [39,41]

(1) General Formulation

x_s : ideal static strength

x_r : residual strength at time t under a loading history s(t)

τ: material age

Introduce normalized variables

$$\bar{x}_r = x_r/\hat{x} , \quad \bar{t} = t/\hat{t} , \quad \bar{x}_s = x_s/\hat{x} \tag{593}$$

$$\frac{d\bar{x}_r}{d\tau} = -\bar{x}_r^{\,1-\alpha_r} , \quad \alpha_r \geq 1 \tag{594}$$

$$\bar{x}_r^{\,\alpha_r} - \bar{x}_s^{\,\alpha_r} = \alpha_r \tau \tag{595}$$

$$d\tau = K(s, \bar{t})\, d\bar{t} \tag{596}$$

Assume

$$K(s, \bar{t}) = \left(\frac{s}{C_1}\right)^{\beta} \bar{t}^{\,\alpha_\ell - 1} \tag{597}$$

Assume a Weibull distribution for static strength

$$R_s(\bar{x}_s) = \exp\left[-\left(\frac{\bar{x}_s}{\bar{x}_{os}}\right)^{\alpha_s} \right] \tag{598}$$

$$\therefore R_r(\bar{x}_r) = R_s(\bar{x}_s) = \exp\left[-\left(\frac{\bar{x}_r^{\,\alpha_r} + \alpha_r \tau}{\bar{x}_{os}^{\,\alpha_r}} \right)^{\alpha_s/\alpha_r} \right] \tag{599}$$

Failure occurs when $x_r(t) = s(t)$.

(2) Tension With Constant Loading Rate

$$s(t) = Lt = (L\hat{t})\,\bar{t} \tag{600}$$

$$\tau = \int_0^{\bar{t}} \left(\frac{L\hat{t}}{C_1}\right)^{\beta} \xi^{\beta + \alpha_\ell - 1}\, d\xi = \frac{(L\hat{t}/C_1)^{\beta}}{\beta + \alpha_\ell}\, \bar{t}^{\,\beta + \alpha_\ell} \tag{601}$$

$$\therefore \bar{x}_r^{\,\alpha_r} = \bar{x}_s^{\,\alpha_r} - \frac{\alpha_r (L\hat{t}/C_1)^{\beta}}{\beta + \alpha_\ell}\, \bar{t}^{\,\beta + \alpha_\ell} \tag{602}$$

At failure, $\bar{x}_r = s(t)/\hat{x} = (L\hat{t}/\hat{x})\,\bar{t}$ (603)

$$\therefore \bar{x}_r^{\alpha_r} = \bar{x}_s^{\alpha_r} - \frac{\alpha_r \hat{x}^{\beta + \alpha_\ell}}{(\beta + \alpha_\ell)(L\hat{t})^{\alpha_\ell} C_1^{\beta}} \bar{x}_r^{\beta + \alpha_\ell} \quad (604)$$

$x_r \longrightarrow x_s$ as $L \longrightarrow \infty$: x_s is the strength under a very high loading rate.

$$R_r(\bar{x}_r) = \exp\left[-\frac{1}{\bar{x}_{os}^{\alpha_s}}\left(\bar{x}_r^{\alpha_r} + \frac{\alpha_r \hat{x}^{\beta + \alpha_\ell}}{(\beta + \alpha_\ell) C_1^{\beta} (L\hat{t})^{\alpha_\ell}} \bar{x}_r^{\beta + \alpha_\ell}\right)^{\alpha_s/\alpha_r}\right] \quad (605)$$

If the second term is negligible compared with the first term, which can happen when, e.g., L is large (fast loading rate), then

$$\bar{x}_r \approx \bar{x}_s \quad (606)$$

$$\therefore R_r(\bar{x}_r) = \exp\left[-\left(\frac{\bar{x}_r}{\bar{x}_{os}}\right)^{\alpha_s}\right] \quad (607)$$

If the first term is negligible, which can happen when, e.g., L is small (slow loading rate), then

$$\therefore R_r(x_r) = \exp\left[-\left(\frac{x_r}{x_{or}}\right)^{(\beta + \alpha_\ell)\alpha_s/\alpha_r}\right] \quad (608)$$

$$x_{or} = \left[\frac{(\hat{t}L)^{\alpha_\ell} (\beta + \alpha_\ell) C_1^{\beta} \bar{x}_{os}^{\alpha_r}}{\alpha_r}\right]^{1/(\beta + \alpha_\ell)} \quad (609)$$

Note that, if $\alpha_\ell = \alpha_r = 1$, then Eq. (608) is equivalent to Eq. (579).

(3) Stress Rupture

$$s(t) = s \quad \text{const.} \quad (610)$$

$$\tau = (s/C_1)^{\beta} \bar{t}^{\alpha_\ell}/\alpha_\ell \quad (611)$$

$$R_r(\bar{x}_r) = \exp\left[-\left(\frac{\bar{x}_r^{\alpha_r} + (\alpha_r/\alpha_\ell)(s/C_1)^{\beta} \bar{t}^{\alpha_\ell}}{\bar{x}_{os}^{\alpha_r}}\right)^{\alpha_s/\alpha_r}\right] \quad (612)$$

$$R_\ell(\bar{t}) = R_r(s/\hat{x}) = \exp\left[-\left(\frac{(s/\hat{x})^{\alpha_r} + (\alpha_r/\alpha_\ell)(s/C_1)^{\beta} \bar{t}^{\alpha_\ell}}{\bar{x}_{os}^{\alpha_r}}\right)^{\alpha_s/\alpha_r}\right] \quad (613)$$

In the fatigue failure region, the first term can be neglected.

$$R_\ell(\bar{t}) = \exp\left[-\left(\frac{\bar{t}}{\bar{t}_o}\right)^{\alpha_\ell \alpha_s/\alpha_r}\right] \tag{614}$$

$$\bar{t}_o\,(s/C_1)^{\beta/\alpha_\ell} = (\bar{x}_{os}^{\alpha_r}\,\alpha_\ell/\alpha_r)^{1/\alpha_\ell} \tag{615}$$

Note that, if $\alpha_\ell = \alpha_r = 1$, then Eq. (614) is equivalent to Eq. (569).

(4) Constant Amplitude Fatigue

x : realistic static strength

$$s(n) = f(S_{max}, R, n) \quad , \tag{616}$$

$$K(s, n) = \left(\frac{S_{max}}{C_1}\right)^\beta n^{\alpha_\ell - 1} \tag{617}$$

$$\bar{x}_r^{\alpha_r} - \bar{x}^{\alpha_r} = -\frac{\alpha_r}{\alpha_\ell}\left(\frac{S_{max}}{C_1}\right)^\beta n^{\alpha_\ell} \tag{618}$$

$$R_r(\bar{x}_r) = \bar{R}_s(\bar{x}_s \mid \bar{S}_{max}) = \exp\left[-\frac{(\bar{x}_r^{\alpha_r} + (\alpha_r/\alpha_\ell)\,(S_{max}/C_1)^\beta n^{\alpha_\ell})^{\alpha_s/\alpha_r}}{\bar{x}_o^{\alpha_s}} + \left(\frac{S_{max}}{\bar{x}_o}\right)^{\alpha_s}\right] \tag{619}$$

$$R_\ell(N) \approx \exp\left[-\left(\frac{N}{N_o}\right)^{\alpha_\ell \alpha_s/\alpha_r}\right] \tag{620}$$

$$N_o(S_{max}/C_1)^{\beta/\alpha_\ell} = (\bar{x}_o^{\alpha_r}\,\alpha_\ell/\alpha_r)^{1/\alpha_\ell} \tag{621}$$

(a) $\alpha_r = \alpha_s$

$$R_r(\bar{x}_r) = \exp\left[-\frac{\bar{x}_r^{\alpha_s} + (\alpha_s/\alpha_\ell)(S_{max}/C_1)^\beta n^{\alpha_\ell}}{\bar{x}_o^{\alpha_s}} + \left(\frac{\bar{S}_{max}}{\bar{x}_o}\right)^{\alpha_s}\right]$$

$$= \exp\left[-\left(\frac{\bar{x}_r}{\bar{x}_o}\right)^{\alpha_s} - \left(\frac{n}{N_o}\right)^{\alpha_\ell} + \left(\frac{S_{max}}{\bar{x}_o}\right)^{\alpha_s}\right] \tag{622}$$

$$R_\ell(N) = \exp\left[-\left(\frac{N}{N_o}\right)^{\alpha_\ell}\right] \tag{623}$$

$$N_o(S_{max}/C_1)^{\beta/\alpha_\ell} = (x_o^{\alpha_s}\alpha_\ell/\alpha_s)^{1/\alpha_\ell} \tag{624}$$

$$\bar{R}_r(x_r|S_{max}) = \exp\left[-\left(\frac{x_r}{x_o}\right)^{\alpha_s} + \left(\frac{S_{max}}{x_o}\right)^{\alpha_s}\right] \tag{625}$$

Note that

$$\bar{R}_r(\bar{S}_{max}) = R_\ell(N) \tag{626}$$

(b) $\alpha_\ell = 1$

$$R_r(\bar{x}_r) = \exp\left[-\frac{\left(\bar{x}_r^{\alpha_r} + \alpha_r(S_{max}/C_1)^\beta n\right)^{\alpha_s/\alpha_r}}{\bar{x}_o^{\alpha_s}} + \left(\frac{S_{max}}{x_o}\right)^{\alpha_s}\right]$$

$$= \exp\left\{-\left[\left(\frac{\bar{x}_r}{\bar{x}_o}\right)^{\alpha_r} + \left(\frac{n}{N_o}\right)\right]^{\alpha_s/\alpha_r} + \left(\frac{S_{max}}{x_o}\right)^{\alpha_s}\right\} \tag{627}$$

$$R_\ell(N) = \exp\left[-\left(\frac{N}{N_o}\right)^{\alpha_s/\alpha_r}\right] \tag{628}$$

$$N_o(S_{max}/C_1)^\beta = \bar{x}_o^{\alpha_r}/\alpha_r \tag{629}$$

Strength and fatigue properties of $[0/45/90/-45_2/90/45/0]_s$ Gr/Ep.

	α_s	x_o, MPa	α_r	α_ℓ	β	C_1, MPa
Model (a)	24.12	487	24.12	1.089	21.68	6.153
Model (b)	24.12	487	22.16	1	19.92	6.231

(5) Cumulative Damage Model

n_1 cycles at S_{max1} followed by n_2 cycles at S_{max2}.

$$\bar{x}_{r1}^{\alpha_r} - \bar{x}^{\alpha_r} = \frac{\alpha_r}{\alpha_\ell}\left(\frac{S_{max1}}{C_1}\right)^\beta n_1^{\alpha_\ell} \tag{630}$$

$$\bar{x}_{r2}^{\alpha_r} - \bar{x}_{r1}^{\alpha_r} = \frac{\alpha_r}{\alpha_\ell}\left(\frac{S_{max2}}{C_1}\right)^\beta n_2^{\alpha_\ell} \tag{631}$$

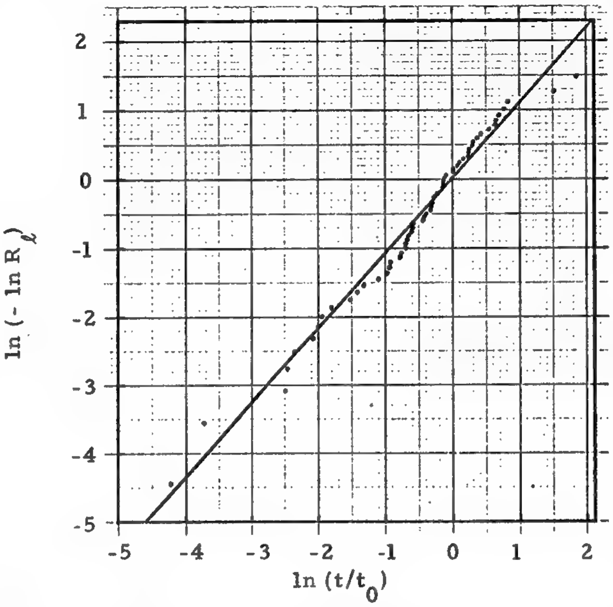


Figure 137 Weibull plot of normalized life data. [32]

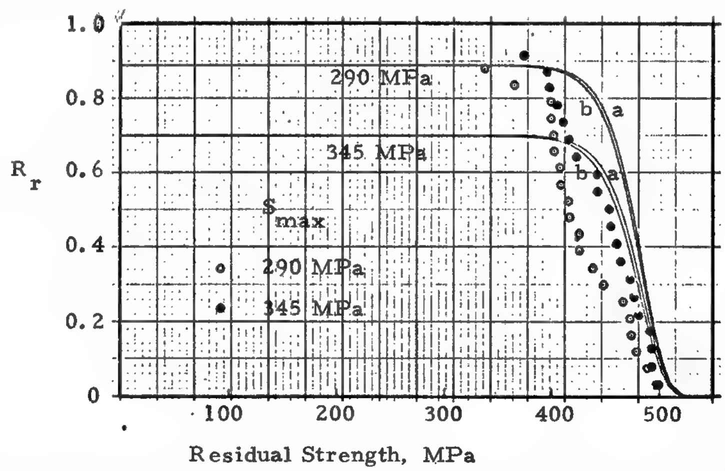


Figure 138 Experimental correlation of analytical models in terms of residual strength. [32]

$$\therefore \; \bar{x}_{r2}^{\alpha_r} = \bar{x}^{\alpha_r} + \frac{\alpha_r}{\alpha_\ell}\left\{\left(\frac{S_{max1}}{C_1}\right)^{\beta} n_1^{\alpha_\ell} + \left(\frac{S_{max2}}{C_1}\right)^{\beta} n_2^{\alpha_\ell}\right\} \tag{632}$$

$$\therefore \; R_r(\bar{x}_{r2}) = \exp\left\{-\frac{\left\{x_{r2}^{\alpha_r} + (\alpha_r/\alpha_\ell)\left[(S_{max1}/C_1)^{\beta} n_1^{\alpha_\ell} + (S_{max2}/C_1)^{\beta} n_2^{\alpha_\ell}\right]\right\}^{\alpha_s/\alpha_r}}{x_o^{\alpha_s}} + \left(\frac{S_{max1}}{x_o}\right)^{\alpha_s}\right\} \tag{633}$$

(a) $\alpha_r = \alpha_s$

$$R_{\ell n1}(N_2) = \exp\left[-\left(\frac{n_1}{N_{o1}}\right)^{\alpha_\ell} - \left(\frac{N_2}{N_{o2}}\right)^{\alpha_\ell}\right] \tag{634}$$

For $R_{\ell n1}(N_2) = 1$

$$\left(\frac{n_1}{N_{o1}}\right)^{\alpha_\ell} + \left(\frac{N_2}{N_{o2}}\right)^{\alpha_\ell} = 1 \tag{635}$$

(b) $\alpha_\ell = 1$

$$R_{\ell n1}(N_2) = \exp\left[-\left(\frac{n_1}{N_{o1}} + \frac{N_2}{N_{o2}}\right)^{\alpha_s/\alpha_r}\right] \tag{636}$$

For $R_{\ell n1}(N_2) = 1$

$$\frac{n_1}{N_{o1}} + \frac{N_2}{N_{o2}} = 1 \tag{637}$$

(c) Experimental correlation

Strength and fatigue properties of $[0_3/\pm 45]$ B/E_p

	α_s	x_o, MPa	α_r	α_ℓ	β	C_1, MPa
Model (a)	24.19	845	24.19	0.4881	20.73	3.989
Model (b)	24.19	845	49.56	1	42.47	3.358

S_{max1}, MPa	N_{o1}	n_1	S_{max2}, MPa	N_{o2}
649	1.3694×10^6	5×10^5	724	1.2454×10^4

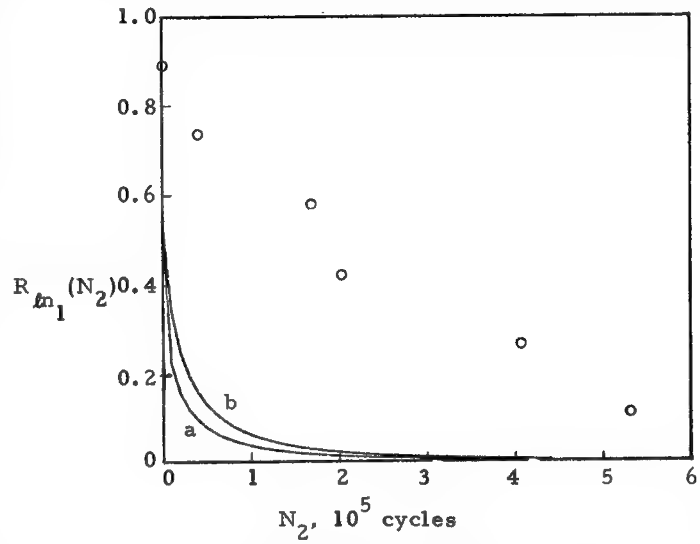


Figure 139 Distribution of N_2: theory and data. [31]

SECTION XII

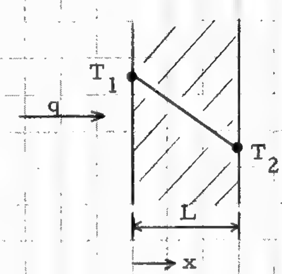
ENVIRONMENTAL EFFECTS ON COMPOSITES

1. HEAT CONDUCTION

a. Formulation

Heat Transfer (temperature effects)

$$q = -K\nabla T \quad (638)$$

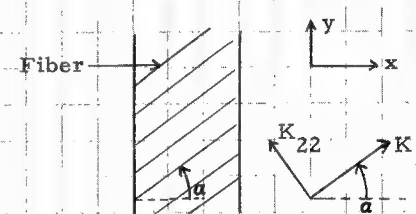


One-dimensional steady

$$q_x = +K_x \frac{T_1 - T_2}{L} \quad (639)$$

Composite Materials

$$K_x = K_{11}\cos^2\alpha + K_{22}\sin^2\alpha \quad (640)$$



Approximations ($v_f < 0.785$)

$$K_{11} = (1-v_f)K_r + v_f K_f \quad (641)$$

$$K_{22} = K_r\left\{(1 - 2\sqrt{v_f/\pi}) + \frac{1}{B_K}\left[\pi - \frac{4}{\sqrt{1 - B_K^2 v_f/\pi}}\tan^{-1}\frac{\sqrt{1-(B_K^2 v_f/\pi)}}{1+\sqrt{B_K^2 v_f/\pi}}\right]\right\} \quad (642)$$

$$B_K \equiv 2\left(\frac{K_m}{K_f} - 1\right) \quad (643)$$

Input: K_r, K_f, v_f, α

Output: K_{11}, K_{22}, K_x

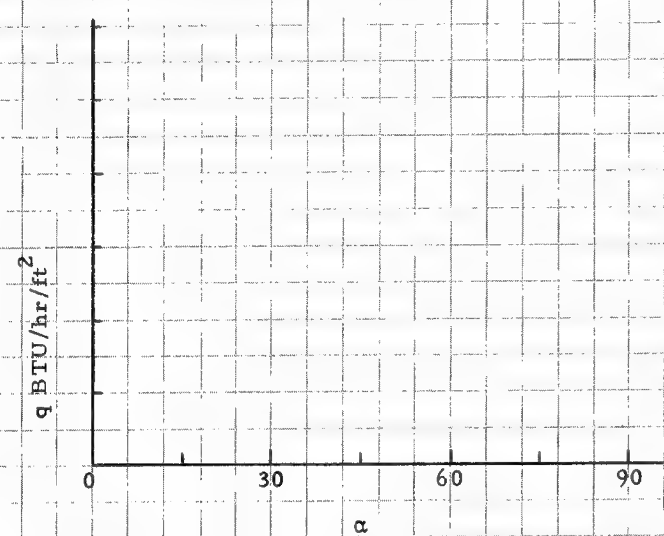
Thermal conductivities of selected materials

K - BTU/hr/ft^2/^{o}F/in; room temperature

Resins	K	Filaments	K
Phenolic	1.4	Carbon fabric	40.0
Phenolic, SC1008	1.5	Glass fabric	7.5
Polyethylene	1.7	Graphite fabric	1000.0
		Quartz fabric	7.5
		Silica fabric	7.5

b. Example

A 0.5-inch thick fiber reinforced graphite epoxy plate is exposed to a $120^{\circ}F$ environment on one side, and to $70^{\circ}F$ on the opposite side. The fiber volume fraction is 68%. Estimate the heat transfer through the plate for the following fiber orientations: α = 0, 30, 45, 60 and 90 degrees.

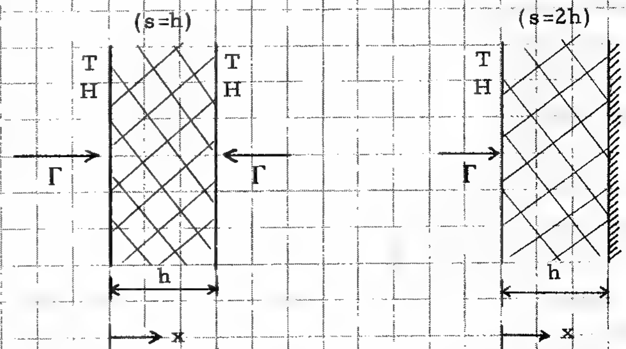


2. MOISTURE DIFFUSION

a. Formulation

Moisture Effects

$$\Gamma = -D\nabla C \tag{644}$$



Steady Bound. Cond.

One Dimensional

Temp Constant

Uniform Initial Distributions

Moisture Distribution:

$$C^* = \frac{C-C_i}{C_m-C_i} = 1 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{(2j+1)}\sin\frac{(2j+1)\pi x}{h}\exp\left\{-\left[(2j+1)^2\pi^2\right]\frac{D_x t}{s^2}\right\} \tag{645}$$

Moisture Content (weight gain):

$$M = \frac{W-W_D}{W_D} = G(M_m - M_i) + M_i \tag{646}$$

$$G \equiv 1 - \frac{8}{\pi^2}\sum_{j=1}^{\infty}\frac{\exp\left[-(2j+1)^2\pi^2\left(\frac{D_x t}{s^2}\right)\right]}{(2j+1)^2} \tag{647}$$

Time to reach 99.9% saturation:

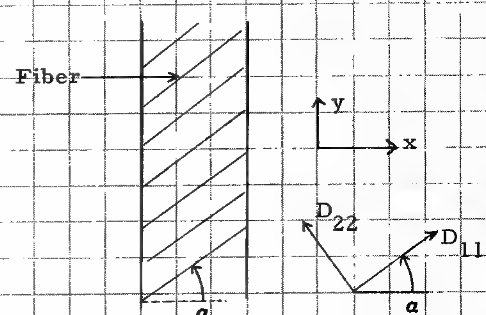
$$t_m = \frac{0.67\, s^2}{D_x} \tag{648}$$

Input: h, M_i, M_m, D_x, t, x (s=h or 2h)

Output: C^*, M, t_m

Maximum Moisture Content:

$$M_m = aH^b \tag{649}$$



Diffusivity:

$$D_x = D_{11}\cos^2\alpha + D_{22}\sin^2\alpha \tag{650}$$

Approximations ($v_f < 0.785$)

$$D_{11} = (1-v_f)D_r + v_f D_f \tag{651}$$

$$D_{22} = D_r\left\{\left(1 - 2\sqrt{\frac{v_f}{\pi}}\right)D_r + \frac{1}{B_D}\left[\pi - \frac{4}{\sqrt{1 - \frac{B_D^2 v_f}{\pi}}}\tan^{-1}\frac{\sqrt{1 - B_D^2 v_f/\pi}}{1 + \sqrt{B_D^2 v_f/\pi}}\right]\right\} \tag{652}$$

$$B_D = 2\left(\frac{D_r}{D_f} - 1\right) \tag{653}$$

For $D_f << D_r$

$$D_x = D_x\left[(1-v_f)\cos^2\alpha + (1 - 2\sqrt{v_f/\pi})\sin^2\alpha\right] \tag{654}$$

Input: D_r, D_f, v_f, α

Output: D_{11}, D_{22}, D_x

b. Examples

(1) Both sides of a 12.5 mm thick Graphite T-300 Fiberite plate are exposed to air at 350°K and 90 percent humidity. The initial moisture concentration is uniform inside the plate. The initial moisture content of the plate is 0.5 percent. The diffusivities: D_{11} and D_{22} are given in the accompanying figure (v_f=0.68). The constants a and b are 0.0014 and 2, respectively.

a) Estimate the time required to reach one percent moisture content
b) Estimate the time required to reach at least 99.9% of the maximum possible moisture content
c) Estimate the maximum possible moisture content inside the material
d) Estimate the moisture content in 5 years
e) Draw the moisture distribution inside the material after 5 years

Perform the calculations for fiber orientations α = 0, 30, 60, and 90 degrees.

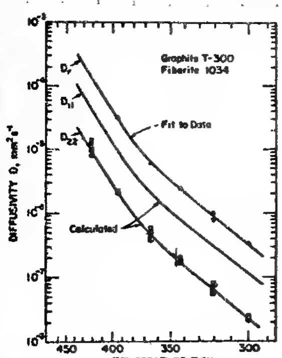


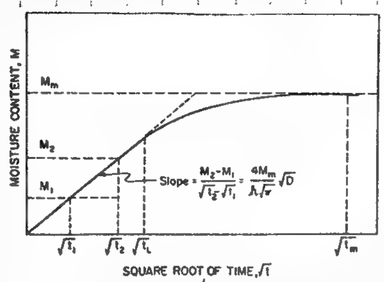
Figure 140 Change of diffusivity with temperature

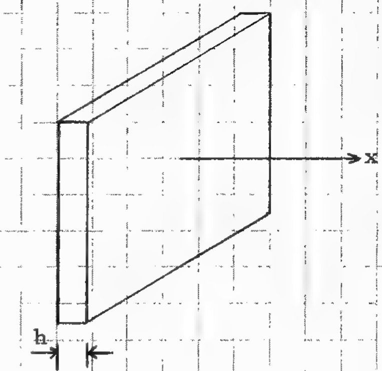
(2) The plate described in the previous example is exposed to air on one side only. The other side of the plate is insulated. Repeat the calculations for this plate.

(3) The initial moisture distribution is uniform inside a 12.5 mm thick Graphite T-300-Fiberite 1034 plate. The initial moisture content of the plate is one percent. The plate is then exposed on both sides to humid air at 333°K and 10% relative humidity. Estimate the

moisture content of the plate after 10 days. The fiber volume fraction is $v_f = 0.68$, the constants a and b are 0.0014 and 2, respectively. The diffusivity of the resin is $D_r = 1.6 \times 10^{-6}$ $mm^2\ s^{-1}$. Perform the calculations for fiber orientations $\alpha = 0, 30, 60,$ and 90 degrees.

c. Tests to Determine M_m and D





Edge effects negligible [42]

$$D_x = \pi \left(\frac{h}{4M_m}\right)^2 \left(\frac{M_2 - M_1}{\sqrt{t_2} - \sqrt{t_1}}\right)^2 \tag{655}$$

for a) accelerated test procedures

b) edge effects corrections

Observations (approximations)

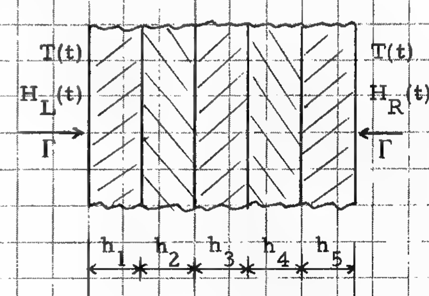
$D \cong f(T)$ (but not M?)

$M_m \cong f(H)$ (but not T?)

$$M_m = aH^b \tag{656}$$

Present information $b = 1 \sim 2$

d. Multilayered-Unsteady Problem



One Dimensional

Temp same on both sides, or

One side insulated

Numerical Solution ("W8GAIN")

Input: $T(t)$, $H_L(t)$, $H_R(t)$

Initial moisture distribution

$D(T)$, $M_m(H)$, $K(T)$, ρ, h (for each layer)

Output: (As a Fn of Time)
- Total Weight Gain
- Weight Gain of Each Layer
- Moisture Distribution in Each Layer

TABLE 70 SUMMARY OF EXPERIMENTAL DATA ON THE EFFECTS OF MOISTURE AND TEMPERATURE ON THE ULTIMATE TENSILE STRENGTH OF COMPOSITES

Composite	Reference	Laminate Lay-Up Orientation						Remarks
		0^o		$\pi/4$		90^o		
		Moist	Temp	Moist	Temp	Moist	Temp	
Thornel 300/ Fiberite 1034	Shen & Springer 1976	L	N	L	N	S	S	
Hercules AS-5/ 3501	Browning et al 1976	N	N	N	N	S	S	
	Verette 1975	N	N	N	-	S	S	Limited data (2-3 points)
	Kerr et al 1975	-	N	-	N	-	S	Two data points for 90^o laminates
	Kim & Whitney 1976	-	-	N	N	-	-	
Thornel 300/ Narmco 5208	Hofer et al 1975	L	L	N	L	S	S	
	Husman 1976	-	-	-	-	S	L	
Modmor II/ Narmco 5206	Hofer et al 1974	N	L	N	L	S	S	
Courtaulds HMS/ Hercules 3002M	Hofer et al 1974	N	N	N	N	S	S	Very scattered data for 90^o laminates
HT-S/ERLA-4617	Browning 1972	-	-	L	S	-	-	Only two data points for temperature
HT-S/ Fiberite X-911	Browning 1972	-	-	N	N	-	-	
HT-S/U.C.C.X-2546	Browning 1972	-	-	L	N	-	-	
PRD 49/ ERLB-4617	Hanson 1972	-	L	-	-	-	-	
HT-S/(8183/137-NDA-BF_3:MEA)	Hertz 1973	-	-	-	-	S	S	
HT-S/Hysol ADX-516	Browning 1972	-	-	N	S	-	-	Only two data points for temperature
Hercules HT-S/ 710 Polyimide	Kerr et al 1975	-	N	-	N	-	N	Only two data points for 90^o laminates

TABLE 70 (Continued)

Composite	Reference	Laminate Lay-Up Orientation						Remarks
		0^o		$\pi/4$		90^o		
		Moist	Temp	Moist	Temp	Moist	Temp	
HT-S/P13N Polyimide	Browning 1972	-	-	-	L	-	-	
Boron/AVCO 5505	Hofer et al 1974	L	N	L	L	S	S	
Boron/Narmco 5505	Kaminski 1973	-	L	-	-	-	S	
	Browning 1972	-	-	N	N	-	-	

(a) N = Negligible effect (b) L = Little effect (<30%) (c) S = Strong effect (>30%)

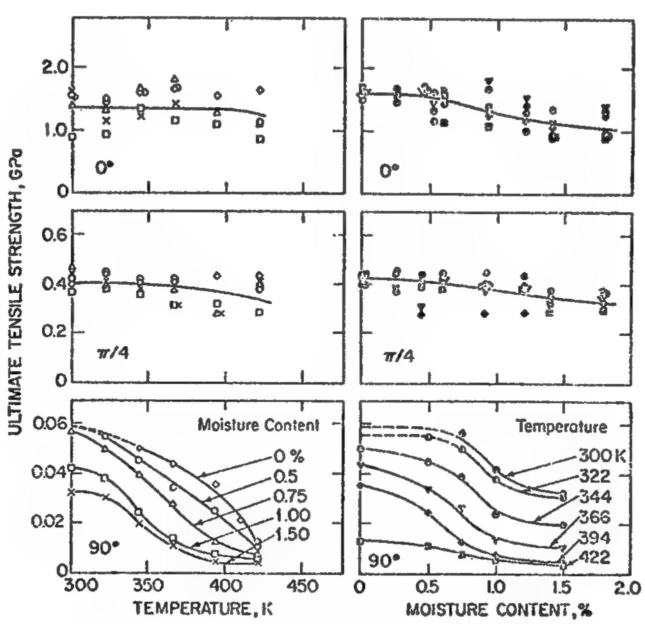


Figure 141 Effects of temperature and moisture on strengths of T300/1034 Gr/Ep (v_f = 0.68).

3. TRANSIENT ANALYSIS

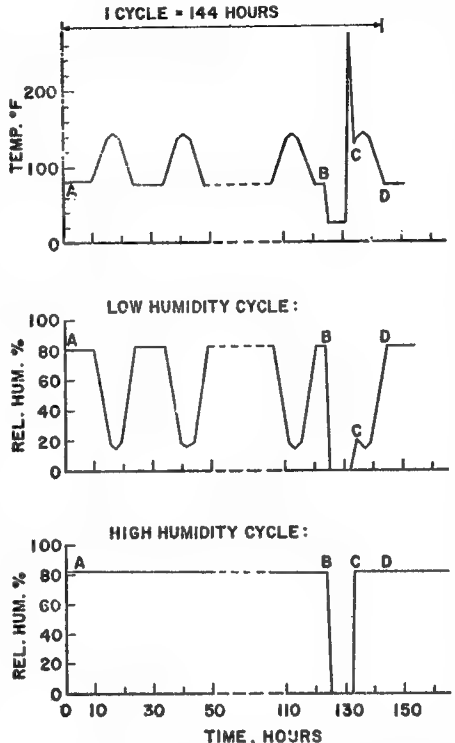


Figure 142 Input to transient analysis.

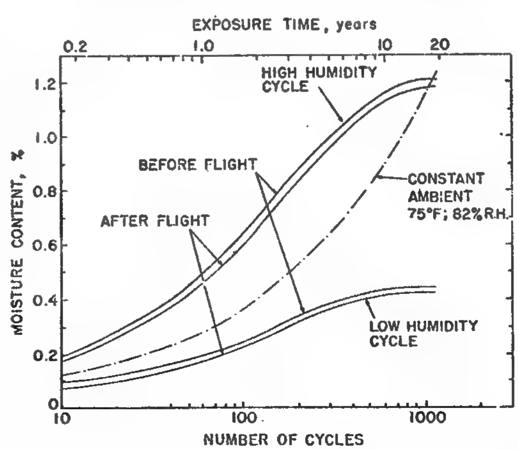


Figure 143 Weight gain over 20 years for 1/2" hybrid.

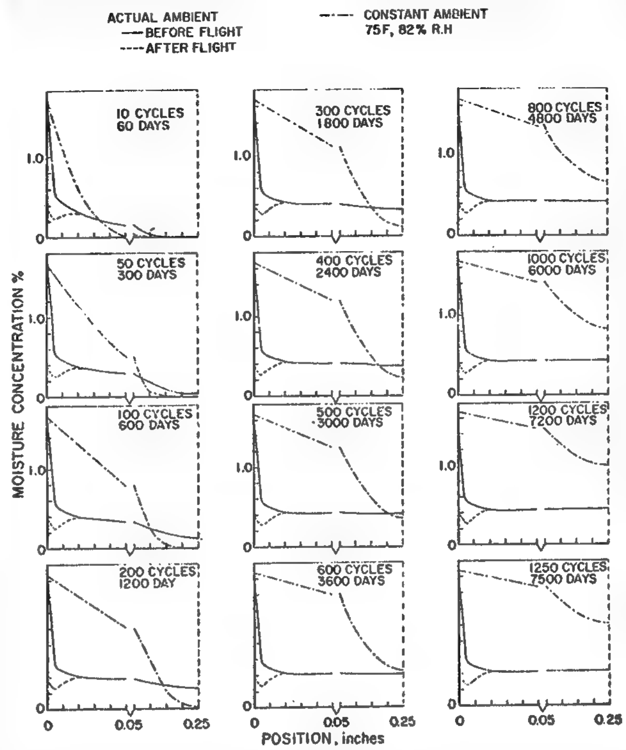


Figure 144 Moisture distribution at boundary layer (1-4 plies) and inside 1/2" hybrid over 20 years - note flat top level at 0.42% after 10 years.

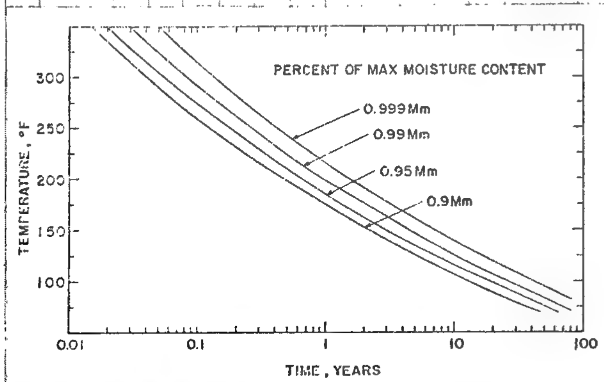


Figure 145 Time required to reach flat top moisture level under constant ambient (insensitive to relative humidity).

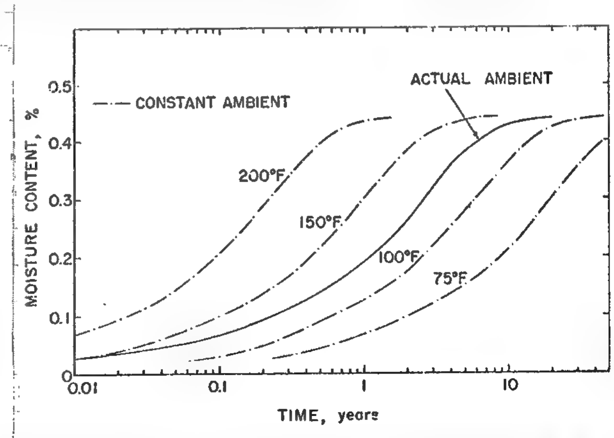


Figure 146 Moisture pickup rates of constant and transient ambients (a guide to accelerated testing).

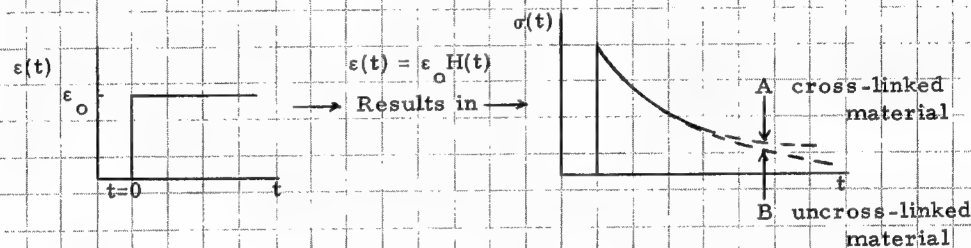
SECTION XIII

TIME DEPENDENCE IN POLYMER DEFORMATION

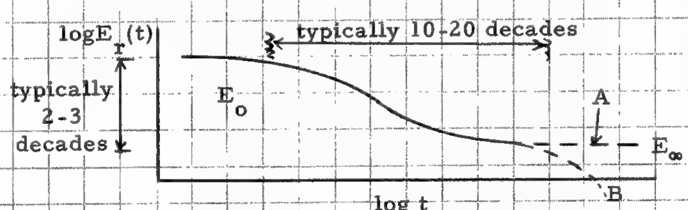
1. LINEARLY VISCOELASTIC CHARACTERIZATION

a. Uniaxial, Homogeneous Deformation; Isotropic Solid

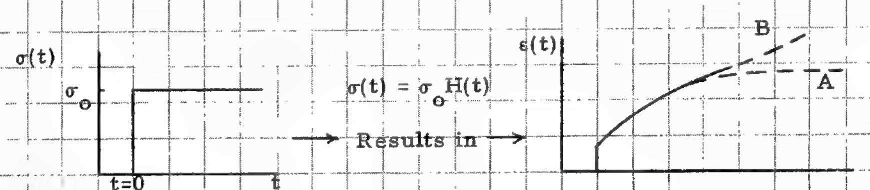
(1) Relaxation behavior (prescribe step strain)



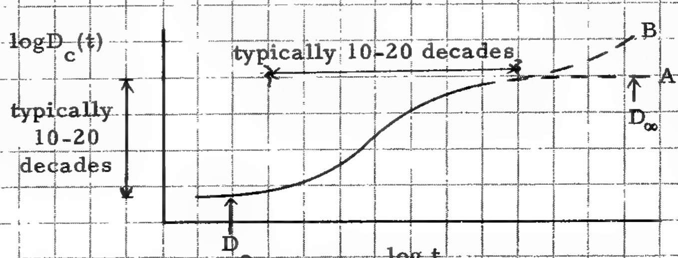
$$\frac{\sigma(t)}{\varepsilon_o} = E_r(t) \text{ ; Relaxation Modulus} \tag{657}$$



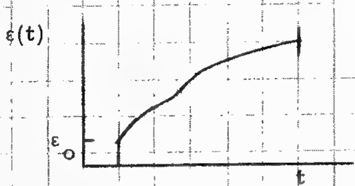
(2) Creep behavior (prescribe step stress)



$$\frac{\varepsilon(t)}{\sigma_o} = D_c(t) \text{ ; Creep Compliance} \tag{658}$$



(3) Non-simple histories



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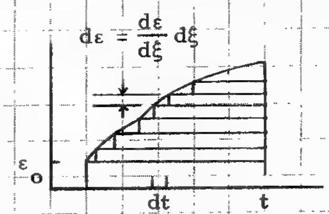
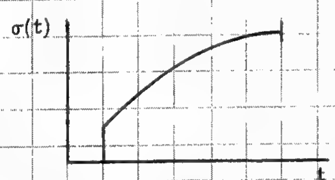


sum
of
small (infinitesimal)
steps

$$\sigma(t) = \varepsilon_o H(t) + \int_{0^+}^{t} E_r(t-\xi)\,\frac{d\varepsilon(\xi)}{d\xi}\,d\xi \qquad \xleftrightarrow{t-\xi} \tag{659}$$

$$\varepsilon(t) = \sigma_o D(t) + \int_{0^+}^{t} D_c(t-\xi)\,\frac{d\sigma(\xi)}{d\xi}\,d\xi \tag{660}$$



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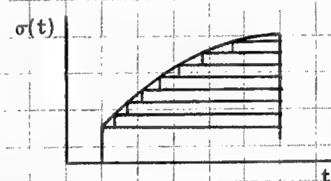


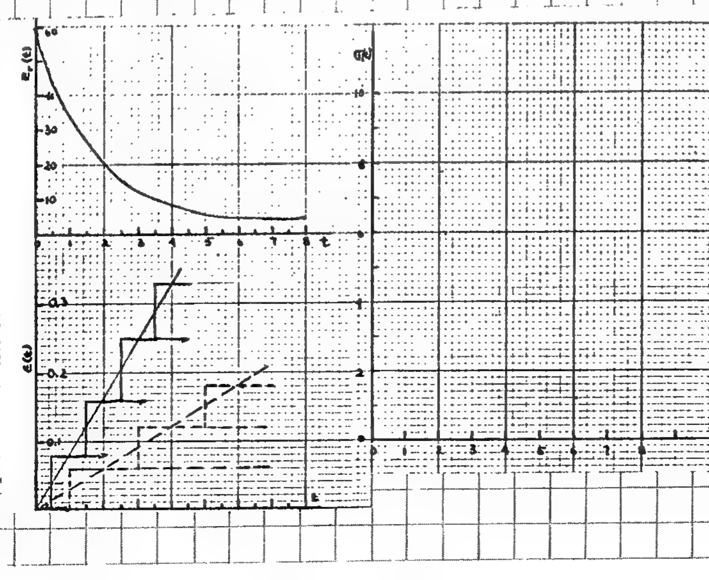
$$\int_{0^+}^{t} E_r(t-\xi)\,\frac{dD(\xi)}{d\xi}\,d\xi = 1 \quad \text{or} \quad \int_{0^+}^{t} E_r(t-\xi)\,D_c(\xi)\,d\xi = t \tag{661}$$

(4) Example

Find the stress response in a uniaxial constant strain rate test

$\varepsilon(t) = R \cdot t$

Approximate graphical construction or analytical $E_r(t) = 4 + 50\,e^{-t/3}$ plot answer



(5) Homogeneous shear deformation (totally analogous)

Relaxation modulus in shear $\mu_r(t)$

Creep compliance in shear $J_c(t)$

Shear stress $$\tau(t) = 2\varepsilon_s \mu_r(t) + 2\int_{0^+}^{t} \mu_r(t-\xi)\frac{d\varepsilon_s}{d\xi}\,d\xi \qquad (662)$$

Shear strain $$2\varepsilon_s(t) = \tau_o J_c(t) + \int_{0^+}^{t} J_c(t-\xi)\frac{d\tau}{d\xi}\,d\xi \qquad (663)$$

(6) Cyclic deformation

$$\varepsilon_s(t) = \varepsilon_{os}\cos\omega t\ (+\,i\varepsilon_{os}\sin\omega t = \varepsilon_{os}e^{i\omega t}) \qquad (664)$$

simple example

$$\mu(t) = \mu_\infty + \mu_1 e^{-t/t_1} \qquad (665)$$

$$\tau(t) = 2\varepsilon_{os}\left[\mu(t) - \mu_\infty\right] - 2\varepsilon_{os}\frac{\omega^2 t_1^2\mu_1}{1+\omega^2 t_1^2}e^{-t/t_1} - 2i\varepsilon_{os}\frac{\omega t_1\mu_1}{1+\omega^2 t_1^2}e^{-t/t_1}$$
$$+\ 2\varepsilon(t)\left\{\mu_\infty + \mu_1\frac{\omega^2 t_1^2}{1+\omega^2 t_1^2} + i\,\mu_1\frac{\omega t_1}{1+\omega^2 t_1^2}\right\} \qquad (666)$$

Complex modulus

$$\mu^*(\omega) = \mu'(\omega) + i\,\mu''(\omega) \qquad (667)$$

$$\mu'(\omega) = \mu_\infty + \mu_1\frac{\omega^2 t_1^2}{1+\omega^2 t_1^2} \qquad (668)$$

$$\mu''(\omega) = \mu_1\frac{\omega t_1}{1+\omega^2 t_1^2} \qquad (669)$$

$$\mu(\omega) \equiv \left|\mu^*(\omega)\right| = \sqrt{(\mu')^2 + (\mu'')^2} \qquad \tan\Delta(\omega) \equiv \frac{\mu''(\omega)}{\mu'(\omega)} \qquad (670)$$

Energy loss per cycle (steady state)

$$\mu^*(\omega)\cdot J^*(\omega) = 1\ ;\ J^* = \underbrace{\frac{\mu'}{\mu^2}}_{J'} - i\,\underbrace{\frac{\mu''}{\mu^2}}_{J''} \qquad (671)$$

$$\tau(t) = \sum_{n=1}^{\infty}\tau_n \sin\frac{2n\pi}{T}t \qquad (672)$$

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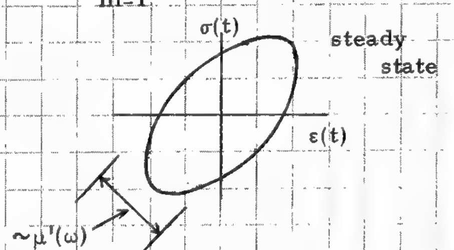
Then

$$\varepsilon_s(t) = \sum_{m=1}^{\infty} \frac{\tau_m}{\mu\left(\frac{2m\pi}{T}\right)} \sin\left[\frac{2m\pi}{T} t + \Delta(2m\pi)\right] \tag{673}$$

$$W/\text{cyle} = \pi \sum_{m=1}^{\infty} m\tau_m^2 J''\left(\frac{2m\pi}{T}\right) = \pi \sum_{m=1}^{\infty} m\tau_m^2 \frac{\mu''\left(\frac{2m\pi}{T}\right)}{\mu^2\left(\frac{2m\pi}{T}\right)} \tag{674}$$

If the strain $\varepsilon_s(t) = \sum_{m=1}^{\infty} \varepsilon_{sm} \sin \frac{2m\pi t}{T}$ is prescribed (675)

$$W/\text{cyle} = \pi \sum_{m=1}^{\infty} m\varepsilon_{sm}^2 \mu''\left(\frac{2m\pi}{T}\right) \tag{676}$$



b. 3-D Stress-Strain Law for Isotropic Solid (Neat Resin)

$$\theta = \sigma_x + \sigma_y + \sigma_z \ , \quad \Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{677}$$

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\theta\delta_{ij} \qquad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\Delta\delta_{ij} \tag{678}$$

$$s_{ij}(t) = 2e_{ij}(o)\,\mu_r(t) + 2\int_{0^+}^{t} \mu_r(t-\xi) \frac{\partial e_{ij}(\xi)}{\partial \xi} d\xi \equiv 2\mu_r(t) * e_{ij}(t) \tag{679}$$

$$\theta(t) = 3\Delta(o)\,K_r(t) + 3\int_{0^+}^{t} K_r(t-\xi) \frac{\partial \Delta(\xi)}{\partial \xi} d\xi \qquad K_r(t) = \text{Bulk relaxation modulus}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{680}$$

$$\begin{Bmatrix} \sigma_x(t) \\ \sigma_y(t) \\ \sigma_z(t) \\ \tau_{xy}(t) \\ \tau_{yz}(t) \\ \tau_{zx}(t) \end{Bmatrix} = \begin{bmatrix} \lambda(t) + 2\mu(t) & \lambda(t) & \lambda(t) & 0 & 0 & 0 \\ \lambda(t) & \lambda(t) + 2\mu(t) & \lambda(t) & 0 & 0 & 0 \\ \lambda(t) & \lambda(t) & \lambda(t) + 2\mu(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu(t) \end{bmatrix} * \begin{Bmatrix} \varepsilon_x(t) \\ \varepsilon_y(t) \\ \varepsilon_z(t) \\ \varepsilon_{xy}(t) \\ \varepsilon_{yz}(t) \\ \varepsilon_{zx}(t) \end{Bmatrix} \tag{681}$$

c. 2-D Orthotropic Relation

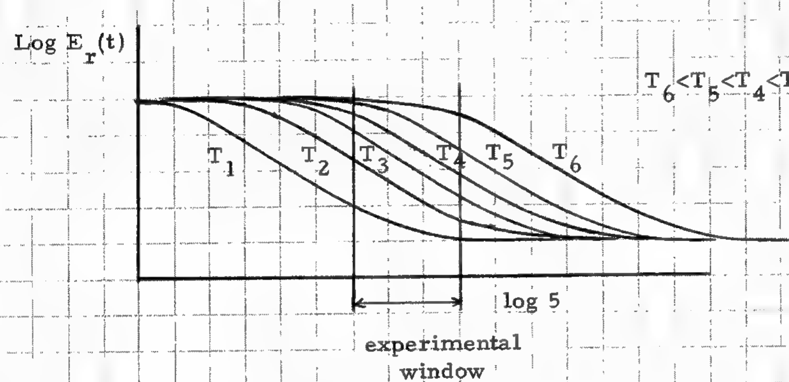
$$\begin{Bmatrix} \sigma_x(t) \\ \sigma_y(t) \\ \tau_{xy}(t) \end{Bmatrix} = \begin{bmatrix} C_{11}(t) & C_{12}(t) & 0 \\ C_{12}(t) & C_{22}(t) & 0 \\ 0 & 0 & C_{44}(t) \end{bmatrix} \ast\ast \begin{Bmatrix} \varepsilon_x(t) \\ \varepsilon_y(t) \\ \varepsilon_{xy}(t) \end{Bmatrix} \quad \text{plane stress} \tag{682}$$

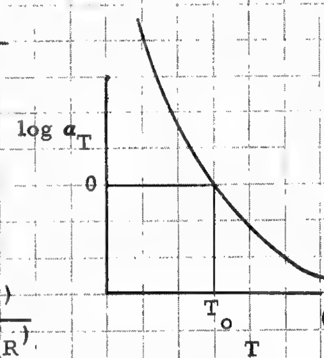
2. EFFECT OF TEMPERATURE ON TIME DEPENDENCE

a. WLF Equation

Log $E_r(t) \sim$ Absolute temperature towards rubbery domain

Short time not well known





WLF Equation $$\text{Log } a_T = \frac{-C_1(T-T_R)}{C_2+(T-T_R)} \tag{683}$$

$C_1 \doteq 8.86$

$C_2 \doteq 102$

$T_R \doteq T_g + 50$

b. Glass Transition Temperature

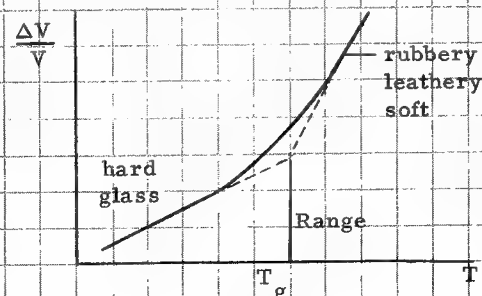


Figure 147 Determination of glass transition temperature.

c. Effect of Water (Plasticizer) on Rheological Behavior

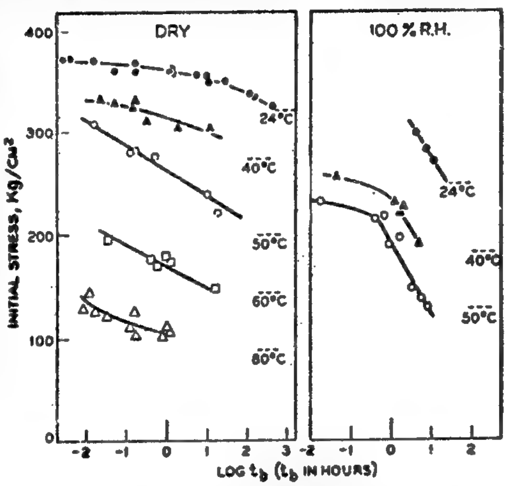


Figure 148 Relationship between stress and breaking time of the HMDA-crosslinked epoxy polymer.

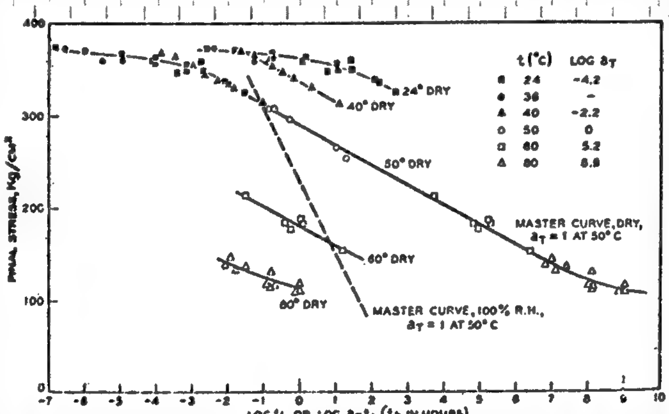


Figure 149 Master curves of σ-log t_b plots of the HMDA polymer.

d. Creep Compliance of Shell 58-68R Epoxy

(Example problem in time-temperature shifting)

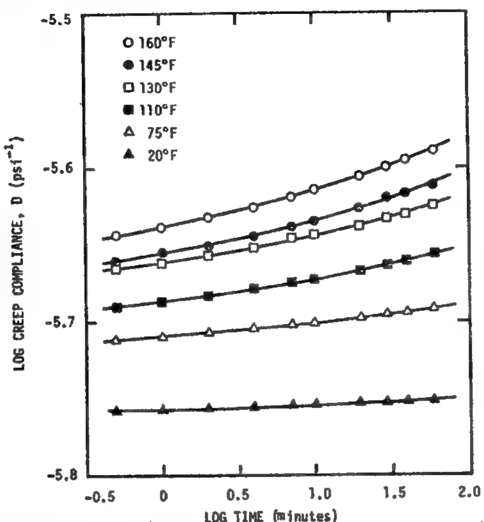


Figure 150 Creep compliance for Shell 58-68R epoxy as a function of temperature.

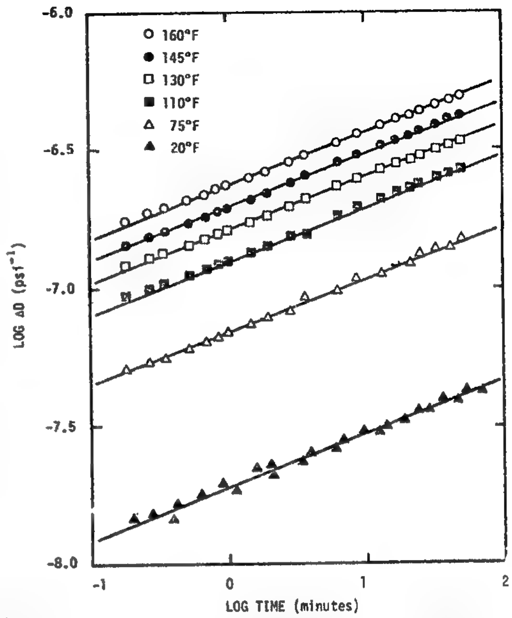


Figure 151 Net creep compliance, ΔD, for Shell 58-68R epoxy at different temperatures.

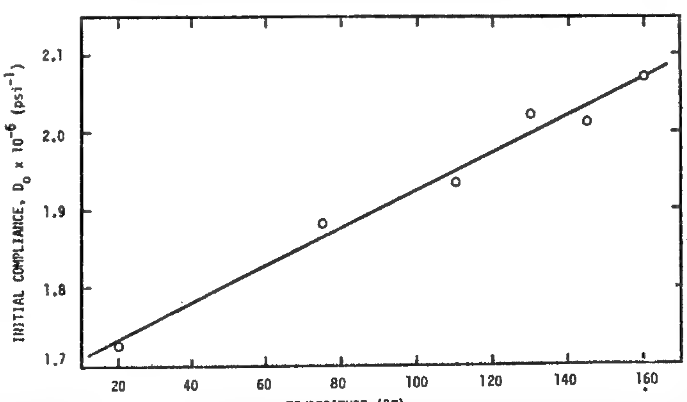


Figure 152 Temperature dependence of initial compliance, D_o, for Shell 58-68R epoxy.

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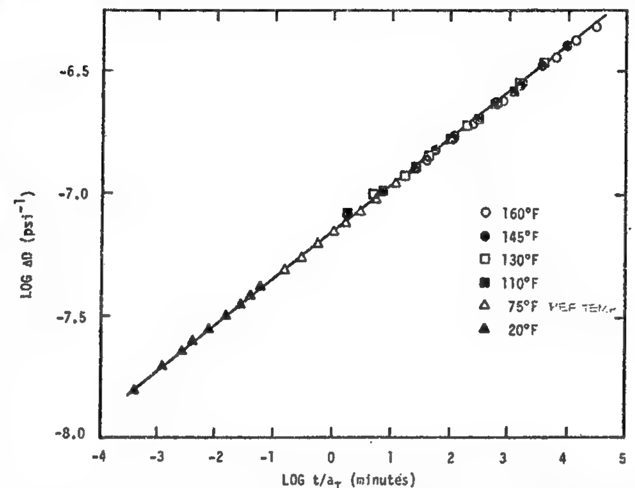

Figure 153 Master curve for net creep compliance, ΔD, for Shell 58-68R epoxy.

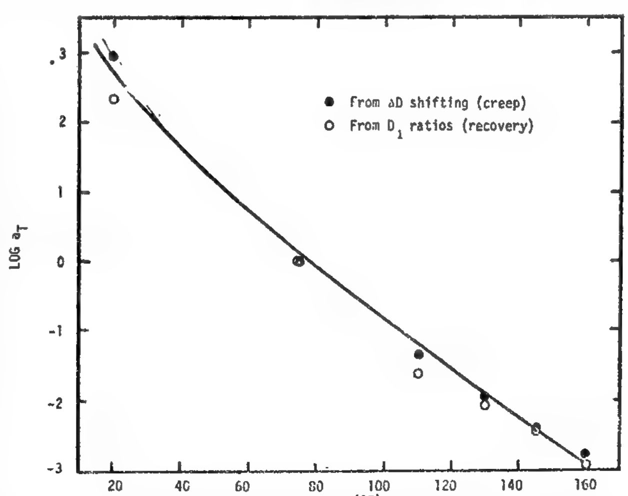

Figure 154 Temperature dependence of the shift factor, a_T, for Shell 58-68R epoxy.

3. PRACTICAL LIMITS ON TIME DOMAIN OF MATERIAL

a. Time Scale of Structural Problem Determined

b. Time Scale of Material that Needs to be Known

c. Principle of Recent Memory

SECTION XIV

APPLICATION OF STATISTICAL THEORIES AND DATA AVERAGING

1. SAMPLE AND PARENT DISTRIBUTIONS

a. Sample Distribution

Results of n random experiments

$$x_1 \leq x_2 \leq x_3 \quad - - - \leq x_n \,,$$

Sample mean $\bar{x}$

$$\bar{x} = \frac{1}{n} \sum_i x_i \tag{684}$$

Sample variance s^2

$$s^2 = \frac{1}{n} \sum_i (x_i - \bar{x})^2 \tag{685}$$

Sample distribution function F(x)

$$P[x \leq x] = F^*(x) = \frac{j-1}{n}, \qquad x_{j-1} \leq x \leq x_j \tag{686}$$

or

$F^*(x_j)$ = j/n upper bound

(j-1)/n lower bound

(j-1/2)/n midpoint

j/(n+1) mean rank

(j-0.3)(n+0.4) median rank

Sample probability density function $f^*(x)$

$$f^*(x) = \frac{dF^*}{dx} \tag{687}$$

b. Parent Distribution

Parent distribution function : F(x)

Parent probability density function : f(x)

Mean of the parent population : μ

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx \tag{688}$$

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Variance of the parent population : σ^2

$$\text{Var}(x) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \tag{689}$$

Let $Y = g(X)$. Then

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx \tag{690}$$

$$\text{Var}[Y] = E[(Y-E[Y])^2] = \int_{-\infty}^{\infty} (g(x) - E[Y])^2 f(x)\,dx \tag{691}$$

We can show that

$$E(\bar{x}) = \mu, \tag{692}$$

$$\text{Var}(\bar{x}) = \sigma^2/n \tag{693}$$

$$E(s^2) = \sigma^2(n-1)/n \tag{694}$$

$$\text{Var}(s^2) = \frac{\mu_4 - \sigma^4}{n} - \frac{2(\mu_4 - 2\sigma^4)}{n^2} + \frac{\mu_4 - 3\sigma^4}{n^3} \tag{695}$$

$$\mu_4 = \int_{-\infty}^{\infty} (x-\mu)^4 f(x)\,dx \tag{696}$$

c. Method of Maximum Likelihood

Problem

Given : A random sample $x_1, x_2, \cdots, x_n$ from the parent distribution whose p.d.f is $f(x;\theta)$, where θ is a parameter.

Find : $\hat{\theta}(x_1, x_2, \cdots, x_n)$ which is a good estimator for θ.

Solution

Consider the function

$$L(x_1, x_2, \cdots, x_n, \theta) = f(x_1;\theta) \cdots f(x_n;\theta) : \tag{697}$$

L represents the probability of obtaining the results $x_1, \cdots, x_n$.

Find value of θ which maximizes L.

Define

$$\mathcal{L} = \log L = \log f(x_1;\theta) + \cdots + \log f(x_n;\theta) \tag{698}$$

Find θ from

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0. \tag{699}$$

d. Strength and Life Distributions

X : strength $\longrightarrow$ $F(x)$: strength distribution

$$F(x) = P\,[\text{strength} \leq x] \tag{700}$$

X : life $\longrightarrow$ $F(x)$: life distribution

$$F(x) = P\,[\text{life} \leq x] \tag{701}$$

Also

$F(x)$: probability of failure

2. WEIBULL AND NORMAL DISTRIBUTIONS

a. Weibull Distribution (Two-Parameter)

(1) Properties

$$F = 1 - \exp\left[-(x/x_o)^{\alpha}\right] \tag{702}$$

α : shape parameter, x_o : scale parameter

Probability density function

$$f = \frac{\alpha}{x_o^{\alpha}} x^{\alpha - 1} \exp\left[-\left(\frac{x}{x_o}\right)^{\alpha}\right] \tag{703}$$

Mean $\mu = x_o \, \Gamma(\frac{1}{\alpha} + 1)$ (704)

Standard deviation $\sigma = x_o \left[\Gamma(\frac{2}{\alpha} + 1) - \Gamma^2(\frac{1}{\alpha} + 1)\right]^{1/2}$ (705)

C.V. (Coefficient of variation) $= \sigma/\mu$

$$C.V. = \left[\frac{\Gamma\left(\frac{2}{\alpha} + 1\right)}{\Gamma^2\left(\frac{1}{\alpha} + 1\right)} - 1\right]^{1/2} \approx \frac{1.2}{\alpha} \text{ or } \left(\frac{1}{\alpha}\right)^{0.94} \tag{706}$$

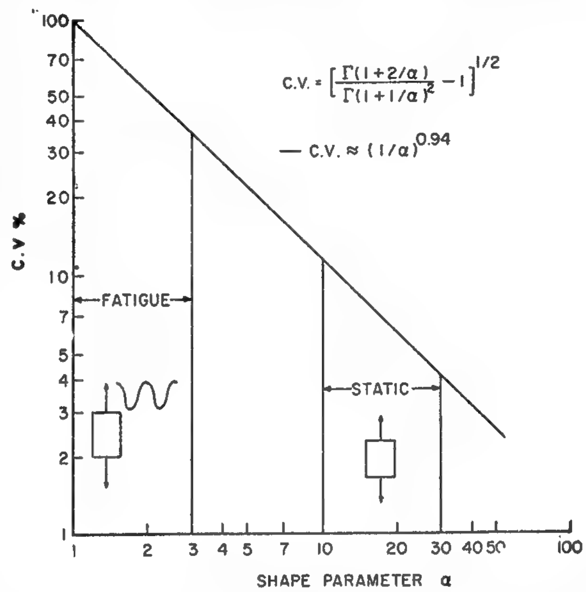

Figure 155 Relation between shape parameter and coefficient of variation.

(2) Estimation of α and X_o

(a) $\hat{\alpha}$ from C.V.

$\hat{x}_o$ from $\hat{\alpha}$ and $\bar{x}$

(b) Maximum likelihood estimate

$$\hat{x}_o = \left(\frac{1}{n}\sum x_i^{\hat{\alpha}}\right)^{1/\hat{\alpha}} \quad (707)$$

$$\hat{\alpha} = \frac{n}{\hat{x}_o^{-\hat{\alpha}} \sum x_i^{\hat{\alpha}} \log x_i - \sum \log x_i} \quad (708)$$

If α is known, then

$$\hat{x}_o = \left(\frac{1}{n}\sum x_i^{\alpha}\right)^{1/\alpha} \quad (709)$$

(c) Linear least squares method (Graphical method)

$$\ell n \left[-\ell n(1-F)\right] = \hat{\alpha}\, \ell n x - \hat{\alpha}\, \ell n \hat{x}_o \quad (710)$$

$$y = ax + b \quad (711)$$

(d) Sample problem

Static strength data for $[0/45/90/-45_2/90/45/0]_s$ T300/934 laminate are as follows:

i	1	2	3	4	5	6	7	8	9	10	11	12
X(MPa)	427	444	445	445	450	455	455	466	469	478	478	481
$F(x_i)$	.028	.067	.106	.146	.185	.224	.264	.303	.343	.382	.421	.461

i	13	14	15	16	17	18	19	20	21	22	23	24	25
x(MPa)	482	482	482	487	487	492	492	495	496	501	503	512	520
$F(x_i)$	.500	.539	.579	.618	.657	.697	.736	.776	.815	.854	.894	.933	.972

Find $\bar{x}$, s, C.V., $\hat{\alpha}$, and $\hat{x}_o$. Use the median rank for F.

$\bar{x}$ = 477 MPa, s = 23 MPa, C.V. = 4.82%

From method (a), $\hat{\alpha}$ = 24.9, $\hat{x}_o$ = 488.

From method (c), $\hat{\alpha}$ = 23.4, $\hat{x}_o$ = 488 (r = 0.983)

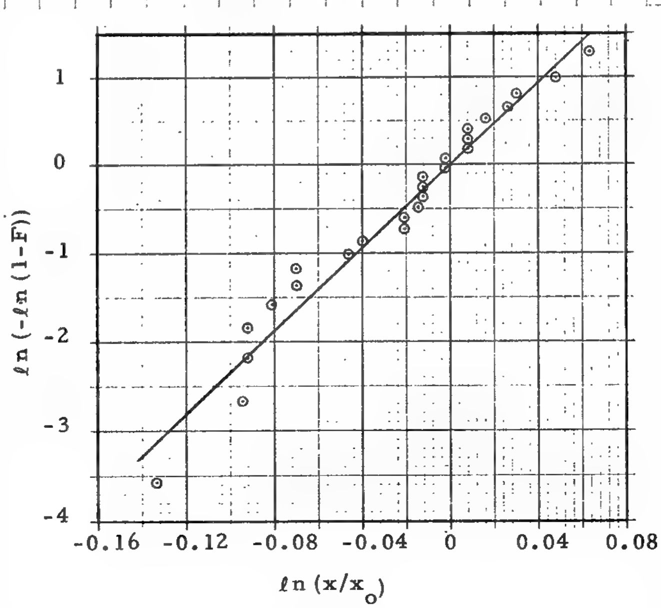

Figure 156 Weibull plot of the strength data. [32]

b. Normal Distribution

(1) Properties

$$F = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right] dt \tag{712}$$

Probability density function

$$f = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{713}$$

Mean : μ

Standard deviation : σ

(2) Estimation of μ and σ

(a) $\hat{\mu} = \bar{x}$, $\hat{\sigma} = s$ (714)

(b) Maximum likelihood estimate

Same as above

(c) Linear least squares method (graphical method)

Define $\frac{x-\mu}{\sigma} = y$ $\qquad \frac{t-\mu}{\sigma} = s$ (715)

$$F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-s^2/2} ds \tag{716}$$

$$x = \sigma y + \mu \tag{717}$$

(d) Sample problem

For the data of the previous sample problem 2.a.(2).(d), determine $\hat{\mu}$ and $\hat{\sigma}$.

From method (a), $\hat{\mu} = \bar{x} = 478$ MPa, $\hat{\sigma} = s = 23$ MPa.

For method (b), first determine

i	1	2	3	4	5	6	7	8	9	10	11	12	13
y	-1.91	-1.50	-1.25	-1.05	-0.90	-0.76	-0.63	-0.52	-0.40	-0.30	-0.20	-0.10	0

i	14	15	16	17	18	19	20	21	22	23	24	25
y	0.10	0.20	0.30	0.40	0.52	0.63	0.76	0.90	1.05	1.25	1.50	1.91

The linear least squares method yields

$\hat{\mu} = 477$ MPa, $\hat{\sigma} = 24.3$ MPa.

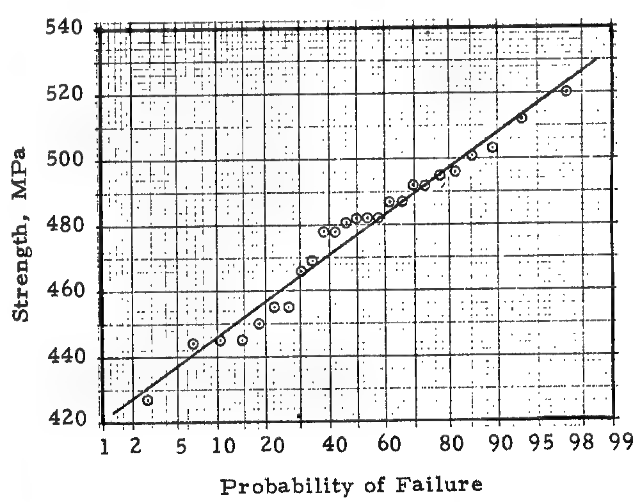

Figure 157 Strength data plotted on normal probability paper.

3. RELIABILITY

a. Definition

The probability of a successful operation of the device in the manner and under the conditions of intended use.

$F(x; \theta_1, \theta_2, \cdots)$: distribution function with parameters $\theta_1, \theta_2, \cdots$.

$x_1 \leq x_2$: limits defining the event : Success.

Reliability function R

$$R = P(x_1 \leq X \leq x_2) = F(x_2; \theta_1, \theta_2, \cdots) - F(x_1; \theta_1, \theta_2, \cdots)$$
$$= R(\theta_1, \theta_2, \cdots; x_1, x_2) \tag{718}$$

b. Reliability for Strength and Life

X : strength

$$R(x) = P[\text{strength} \geq x] = 1 - F(x) \tag{719}$$

X : life

$$R(x) = P[\text{life} \geq x] = 1 - F(x) \tag{720}$$

c. Estimation of R

$$R^* = R(\theta_1^*, \theta_2^*, \cdots; x_1, x_2) \tag{721}$$

4. DESIGN ALLOWABLES

a. Definition

"A" allowable x_A: Probability is 95% that at least 99% of the distribution will be contained within the interval (x_A, ∞).

"B" allowable x_B: Probability is 95% that at least 90% of the distribution will be contained within the interval (x_B, ∞).

b. Weibull Distribution

Suppose α is known.

$\dfrac{2n\,\hat{x}_o^{\alpha}}{x_o^{\alpha}}$ has the χ^2 distribution with 2n degrees of freedom:

$$P\left[\frac{2n\hat{x}_o^{\alpha}}{x_o^{\alpha}} \leq y\right] = \frac{1}{2^n\,\Gamma(n)} \int_o^y t^{n-1} e^{-t/2} dt \tag{722}$$

Given the confidence level γ, let $\chi^2_{2n;\gamma}$ be defined by

$$P\left[\chi^2_{2n} \leq \chi^2_{2n;\gamma}\right] = \gamma \tag{723}$$

$$\therefore P\left[\frac{2n\hat{x}_o^{\alpha}}{x_o^{\alpha}} \leq \chi^2_{2n;\gamma}\right] = \gamma \tag{724}$$

$$= P\left[\frac{2n\hat{x}_o^{\alpha}}{\chi^2_{2n;\gamma}} \leq x_o^{\alpha}\right] \tag{725}$$

Lower confidence limit of $\hat{x}_o$ with confidence level γ: $\hat{x}_\gamma$

$$\hat{x}_\gamma = \left(\frac{2n}{\chi^2_{2n;\gamma}}\right)^{1/\alpha} \hat{x}_o \tag{726}$$

Since $x_{o1} < x_{o2}$ implies $R_1 < R_2$, the lower confidence limit for R, $\hat{R}_\gamma$, is

$$\hat{R}_\gamma = \exp\left[-\left(\frac{x}{\hat{x}_\gamma}\right)^{\alpha}\right] \tag{727}$$

For "A" allowable x_A	$\hat{R}_\gamma = 0.99$	$\gamma = 0.95$
For "B" allowable x_B	$\hat{R}_\gamma = 0.90$	$\gamma = 0.95$

$$x_{A,B} = [-2n\ell n\hat{R}_\gamma / X^2_{2n;\gamma}]^{1/\alpha} \hat{x}_o \quad (728)$$

$$\hat{x}_o = [\frac{1}{n} \sum_{i=1}^{n} x_i^{\hat{\alpha}}]^{1/\alpha} \quad (729)$$

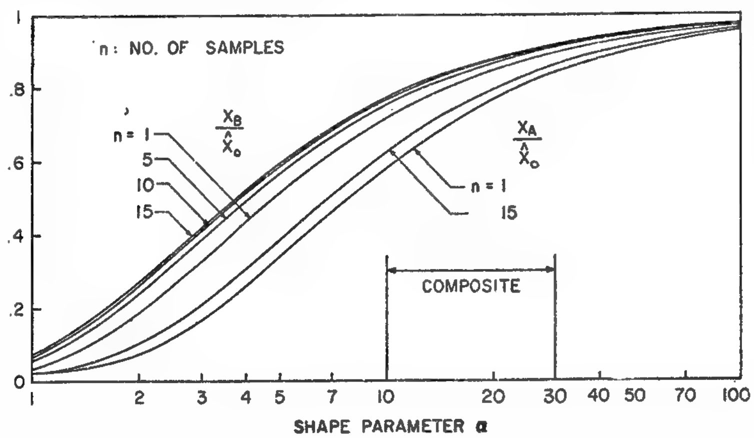

Figure 158 Relation between "A" or "B" allowable and shape parameter.

c. Normal Distribution

$$\bar{x} - Ks', \quad s' = s[n/(n-1)]^{1/2}, \quad \bar{x} = \frac{1}{n}\sum_i x_i, \quad s = [\frac{1}{n}\sum_i (x_i - \bar{x})^2]^{1/2} \quad (730)$$

K is chosen in such a way that the probability is γ that at least a fraction R of the distribution will be contained within the interval between $\bar{x} - Ks'$ and ∞.

For "A" allowable	$R = 0.99$	$\gamma = 0.95$
For "B" allowable	$R = 0.90$	$\gamma = 0.95$

$$x_A = \bar{x} - K_A s', \quad x_B = \bar{x} - K_B s' \quad (731)$$

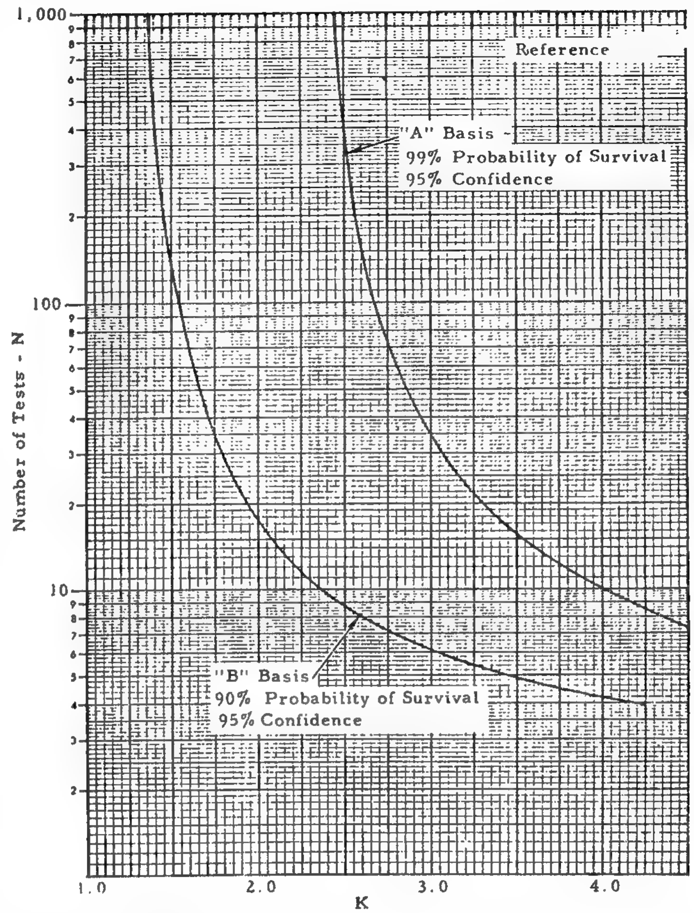

Figure 159 One-sided tolerance factors for normal distribution, K_A and K_B.

d. Sample problem

For the data of sample problem 2.a(2)(d), determine the "A" and "B" allowables.

For the Weibull distribution assume $\hat{\alpha} = 23.4$

Solution

For the Weibull distribution

$$\hat{x}_o = \left[\frac{1}{25} \sum_{i=1}^{25} x_i^{23.4} \right]^{1/23.4} = 488$$

From χ^2 distribution table

$$\chi^2_{50;0.95} = 67.5$$

$$\therefore x_A = [-50 \, \ell n \, 0.99/67.5]^{1/23.4} \quad 488 = 396 \text{ MPa}$$

$$x_B = [-50 \, \ell n \, 0.90/67.5]^{1/23.4} \quad 488 = 438 \text{ MPa}$$

For the normal distribution

$$\bar{x} = 478 \,, \quad s' = 23.5$$

From the figure

$$K_A = 3.15 \qquad K_B = 1.85$$

$$\therefore x_A = 478 - 3.15 \times 23.5 = 404 \text{ MPa}$$

$$K_B = 478 - 1.85 \times 23.5 = 435 \text{ MPa}$$

5. STATISTICAL INTERPRETATION OF FAILURE PROCESS

a. Typical Failure Process

Failure rate at time t : the probability of failure per unit time after having survived to time t.

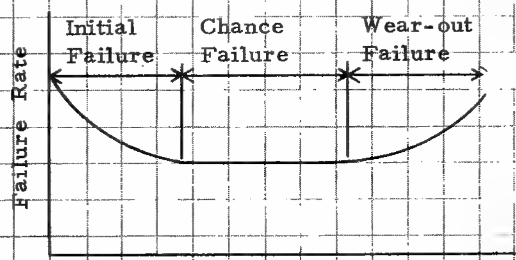

Figure 160 Typical failure rate vs. time.

Initial failure period : Break-in period
Failure due to initial defect or weakness.

Chance failure period : Failure due to unusually severe, unpredictable, and/or unavoidable environmental conditions.

Wear-out failure period: Failure due to fatigue, aging.

b. Mathematical Representation of Failure Rate

$\lambda(t)$: failure rate, hazard function, risk function

$f(t)dt$: proportion of the initial population, which will fail in the time interval $(t, t+dt)$

$R(t) = 1 - F(t)$: proportion of the initial population, which have survived to time t.

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{1}{1-F}\frac{dF}{dt} = -\frac{1}{R}\frac{dR}{dt} \quad (732)$$

$$\frac{dR}{R} = -\lambda(t)dt \qquad \ln R = -\int \lambda dt \quad (733)$$

$$\therefore \frac{R(t)}{R(o)} = \exp\left[-\int_o^t \lambda(\xi)d\xi\right] \quad (734)$$

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$$\frac{1-F(t)}{1-F(o)} = \exp\left[- \int_o^t (\xi) d\xi \right] \qquad (735)$$

$$f(t) = (1-F(o))\ \lambda(t)\ \exp\left[- \int_o^t \lambda(\xi) d\xi \right] \qquad (736)$$

For Weibull distribution

$$\lambda(t) = \frac{\alpha}{t_o^{\alpha}}\ t^{\alpha-1} \qquad (737)$$

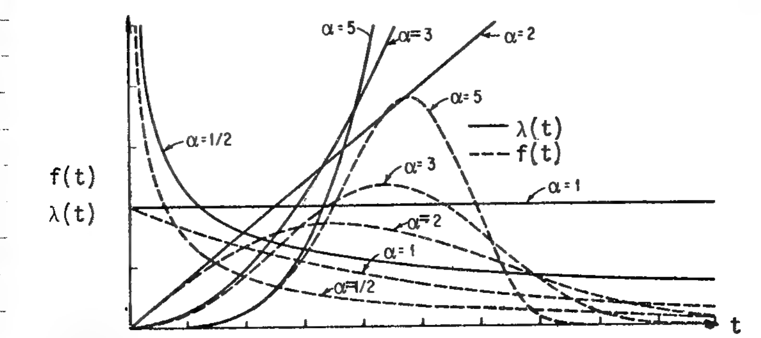


Figure 161 Failure rate and probability density for Weibull distribution.

For normal distribution

$$\lambda(t) = \frac{\exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2 \right]}{\int_t^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\xi-\mu}{\sigma}\right)^2 \right] d\xi} \qquad (738)$$

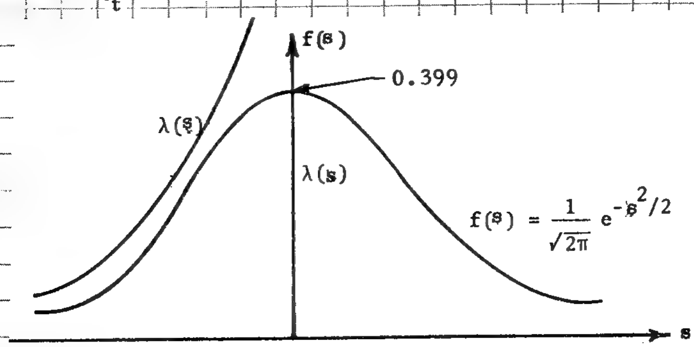


Figure 162 Failure rate and probability density for normal distribution.

c. Concept of Failure Potential

$$\lambda(t) = \frac{d\psi}{dt} \qquad \psi : \text{ failure potential} \tag{739}$$

Introduce

τ : material "age"

τ depends on the loading history $s(\xi)$ through

$$\tau = \int_0^t K(s(\xi))\,d\xi \tag{740}$$

K : breakdown rule

$$\lambda(t) = \frac{d\psi}{d\tau}\,\frac{d\tau}{dt} \tag{741}$$

$$\therefore R(t) = \exp\left[-\psi(\tau)\right] \tag{742}$$

6. SIZE EFFECT

a. Simple Chain or Series Model

Consider a chain consisting of N identical links

$R_o(\sigma)$: reliability of link, i.e., probability of strength exceeding σ.

$R(\sigma)$: reliability of chain

σ is the same in all the links.

The chain fails if at least one link fails.

$$R = R_o^N$$

$$\therefore F = 1 - R = 1 - R_o^N = 1-(1 - F_o)^N . \quad (743)$$

If $R_o = \exp\left[-(\sigma/\sigma_o)^\alpha\right]$, then (744)

$$R(\sigma) = \exp\left[-N(\sigma/\sigma_o)^\alpha\right] = \exp\left[-(\sigma/\sigma_{oN})^\alpha\right] , \quad (745)$$

$$\sigma_{oN} = \sigma_o N^{-1/\alpha} \quad (746)$$

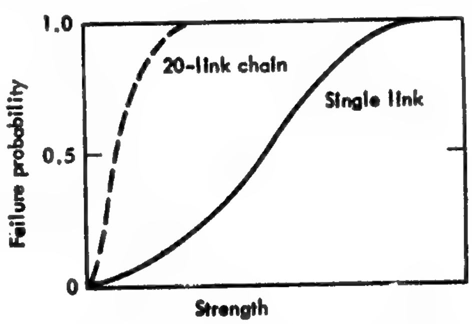


Figure 163 Comparison in probability of failure between single link and 20 - link chain.

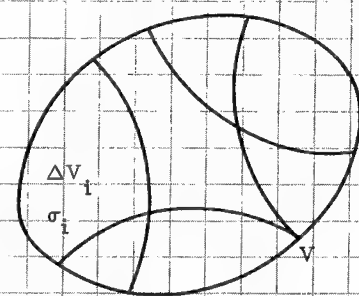
b. General Chain Model

Homogeneous material : uniformly distributed flaws

σ_i is uniform in each element.

$$R_i = \left[R_o(\sigma_i)\right]^{\Delta V_i} \tag{747}$$

$$R = \prod_{i=1}^{N} \left[R_o(\sigma_i)\right]^{\Delta V_i} \tag{748}$$



$$\ell n R = \sum_{i=1}^{N} \Delta V_i \,\ell n \left[R_o(\sigma_i)\right] \tag{749}$$

$$\ell n R = \int_V \ell n \left[R_o(\sigma)\right] dV \tag{750}$$

$$\therefore R = \exp\left[\int_V \ell n \left[R_o(\sigma)\right] dV\right] \tag{751}$$

For Weibull distribution

$$R_o(\sigma) = \exp\left[-(\sigma/\sigma_o)^{\alpha}\right] \tag{752}$$

$$\ell n \left[R_o(\sigma)\right] = -(\sigma/\sigma_o)^{\alpha} \tag{753}$$

$$R = \exp\left[-\int_V (\sigma/\sigma_o)^{\alpha} dV\right] \tag{754}$$

R is the probability of surviving the stress distribution σ.

Let

$$\sigma = X f(\underset{\sim}{x}) \qquad \underset{\sim}{x}: \text{ position vector within the body.} \tag{755}$$

X: reference stress

Then

$$R(X) = \exp\left[-(X/X_o)^{\alpha}\right] \tag{756}$$

$$X_o = \sigma_o \left[\int_V f^{\alpha}(\underset{\sim}{x})\, dV\right]^{-1/\alpha} \quad ; \quad \text{characteristic strength} \tag{757}$$

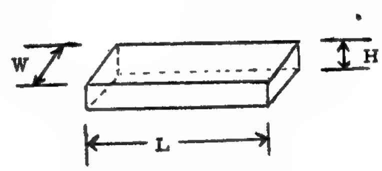
TABLE 71 COMPARISON AMONG CHARACTERISTIC STRENGTHS UNDER DIFFERENT TEST METHODS

	3-pt. flexure (center point loading)	4-pt. flexure (quarter point loading)
Volume	$\frac{(X_o)_f}{(X_o)_t} = \left[2(a+1)^2 \frac{V_t}{V_f}\right]^{\frac{1}{a}}$	$\frac{(X_o)_f}{(X_o)_t} = \left[\frac{4(a+1)^2}{a+2} \frac{V_t}{V_f}\right]^{\frac{1}{a}}$
Surface	$\frac{(X_o)_f}{(X_o)_t} = \left[\frac{(a+1)A_t}{A_f + B_f/(a+1)}\right]^{\frac{1}{a}}$	$\frac{(X_o)_f}{(X_o)_t} = \left[\frac{(a+1)^2 A_t}{B_f/2 + (a+1)(A_f + B_f)/2 + (a+1)^2 A_f/2}\right]^{\frac{1}{a}}$
Edge	$\frac{(X_o)_f}{(X_o)_t} = \left[(a+1) \frac{\ell_t}{\ell_f}\right]^{\frac{1}{a}}$	$\frac{(X_o)_f}{(X_o)_t} = \left[\frac{a+1}{a+2} \frac{\ell_t}{\ell_f/2}\right]^{\frac{1}{a}}$

$V_t = WLH$, $V_f = WLH$

$A_t = 2L(W+H)$, $A_f = WL$, $B_f = LH$

$\ell_t = 4L$, $\ell_f = 2L$



(1) Sample Problem

From tensile coupon tests X_o and α are found to be

$$X_o = 134 \text{ MPa}, \quad \alpha = 4.126.$$

What are the expected values of X_o and α in 3-pt and 4-pt flexure tests?

Specimen dimensions are:

	W mm	L mm	H mm
tension	13	50	4
3-pt flexure	25	64	4
4-pt flexure	25	64	4

Use the volume model in Table 71 .

Solution

$$V_t = 2600 \text{ mm}^2, \quad V_{f3} = 5400 \text{ mm}^2$$

$$V_{f4} = 5400 \text{ mm}^2$$

$$(X_o)_{f3} = 293 \text{ MPa}, \quad (X_o)_{f4} = 224 \text{ MPa}$$

c. Size Effect in Fatigue

For the representative element

$R_o(t \mid \sigma_i)$: Probability of surviving t when subjected to a loading characterized by σ_i.

For the entire body

$$R(t \mid \sigma) = \exp\left[\int_V \ell n \left[R_o(t \mid \sigma)\right] dV\right] \tag{758}$$

If σ is uniform throughout the body, and

$$R_o(t \mid \sigma) = R_o(t) = \exp\left[-(t/t_o)^\beta\right], \tag{759}$$

then

$$R(t) = \exp\left[-V\,(t/t_o)^\beta\right] \tag{760}$$

(1) Sample Problem

The life distribution of a tensile coupon of unit volume subjected to a constant stress σ is given by

$$R_o(t|\sigma) = \exp\left[-\left(\frac{t}{t_o}\right)^{\beta}\left(\frac{\sigma}{\sigma_o}\right)^{\alpha}\right],$$

Determine the life distribution under a three-point bend stress rupture. Use the volume model in Table 71.

Solution

$$\ell n\left[R_o(t|\sigma)\right] = -\left(\frac{t}{t_o}\right)^{\alpha}\left(\frac{\sigma}{\sigma_o}\right)^{\beta}$$

$$R(t|s) = \exp\left[-\left(\frac{t}{t_o}\right)^{\alpha}\int_V\left(\frac{\sigma}{\sigma_o}\right)^{\beta} dV\right]$$

From Table 71,

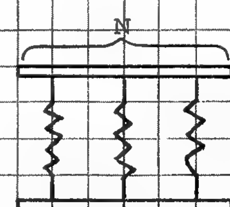
$$\int_V\left(\frac{\sigma}{\sigma_o}\right)^{\beta} dV = \left(\frac{s}{s_o}\right)^{\beta}, \quad s_o = \sigma_o\left[2(\beta+1)^2/V_{\ell}\right]^{1/\beta}$$

$$\therefore R(t|s) = \exp\left[-\left(\frac{t}{t_o}\right)^{\alpha}\left(\frac{s}{s_o}\right)^{\beta}\right]$$

d. Parallel Model

Definition

The system fails when and only when all subsystems fail.



Probability of failure of the system

$$F = F_o^N \tag{761}$$

System reliability

$$R = 1-F = 1-F_o^N \tag{762}$$

(1) Sample Problem

Determine the reliability of a parallel system of 10 bars under a strain controlled test. Assume that the probability of failure of each element is

$F_o = 1 - \exp - (e/0.6)^3$, where e is in %.

Solution

Since e is the same in all bars, and since e is controlled, failure of one bar does not affect the strain in the other bars. Therefore,

$$F = F_o^N = \left[1 - \exp\left[-(e/0.6)\right]^3\right]^{10}$$

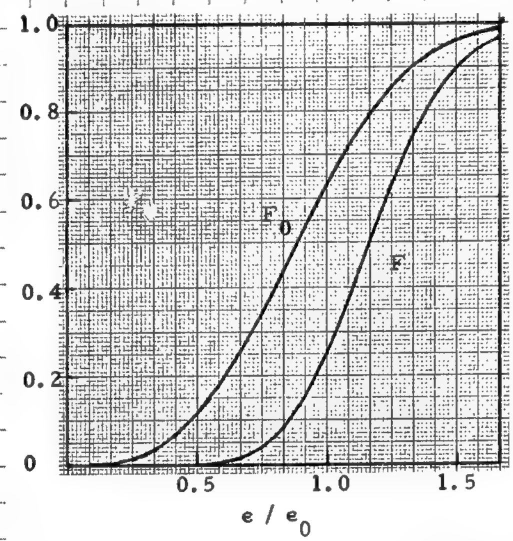


Figure 164 Comparison between F_o and F. System reliability is improved over element reliability.

7. DATA AVERAGING

a. Compliances

Invariants : independent of fiber orientation

Measure : S'_{11}, S'_{22}, S'_{12}, S'_{66}, S'_{16}, S'_{26}

Calculate: I_1, I_2, R_1, R_2

$$I_1 = (S'_{11} + S'_{22} + 2S'_{12}) / 4$$

$$I_2 = (S'_{11} + S'_{22} - 2S'_{12} + S'_{66}) / 8 \tag{763}$$

$$R_1 = [(-S'_{11} + S'_{22})^2 + (S'_{16} + S'_{26})^2]^{1/2} / 2$$

$$R_2 = [(S'_{11} + S'_{22} - 2S'_{12} - S'_{66})^2 + 4(S'_{26} - S'_{16})^2]^{1/2} / 8$$

Calculate average values : $\bar{I}_1$, $\bar{I}_2$, $\bar{R}_1$, $\bar{R}_2$

Average compliances :

$$\bar{S}_{11} = \bar{I}_1 + \bar{I}_2 - \bar{R}_1 - \bar{R}_2$$

$$\bar{S}_{22} = \bar{I}_1 + \bar{I}_2 + \bar{R}_1 - \bar{R}_2 \tag{764}$$

$$\bar{S}_{12} = \bar{I}_1 - \bar{I}_2 + \bar{R}_2$$

$$\bar{S}_{66} = 4\bar{I}_2 + 4\bar{R}_2$$

(1) Sample Problem

Test data for compliance of unidirectional composite

Off-axis Angle	S'_{11}	S'_{12}	S'_{16}	S'_{22}	S'_{26}	S'_{66}
0°	.059	-.028	.0	.892	.0	1.380
15°	.115	-.097	.318	.864	.078	1.292
30°	.308	-.085	.381	.661	.206	1.055
45°	.496	-.126	.390	.496	.338	.896

Determine S_{11}, S_{12}, S_{22}, S_{66}.

Solution

Off-axis Angle	I_1	I_2	R_1	R_2
0^{o}	.2238	.2984	.4165	.0466
15^{o}	.1963	.3081	.4236	.0618
30^{o}	.1998	.2743	.3425	.0450
45^{o}	.1850	.2675	.3640	.0454
Ave.	.2012	.2871	.3867	.0497

$\bar{S}_{11} = .0519$ $\bar{S}_{12} = -.0362$ $\bar{S}_{22} = .8252$ $\bar{S}_{66} = 1.3472$

b. Strength Under Combined Loading

Failure function f

$$f(\sigma_i) = F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 = 1 \qquad (765)$$

Determine : F_1, F_2, F_{11}, F_{12}, F_{22}, F_{66}

Measure σ_1, σ_2, σ_6 at failure.

$$\begin{bmatrix} \sigma_1^{(1)} & \sigma_2^{(1)} & \sigma_1^{(1)^2} & \sigma_1^{(1)}\sigma_2^{(1)} & \sigma_2^{(1)^2} & \sigma_6^{(1)^2} \\ \sigma_1^{(2)} & \sigma_2^{(2)} & \sigma_1^{(2)^2} & \sigma_1^{(2)}\sigma_2^{(2)} & \sigma_2^{(2)^2} & \sigma_6^{(2)^2} \\ - & - & - & - & & \\ & & & & & \\ - & - & - & - & & \\ \sigma_1^{(n)} & \sigma_2^{(n)} & \sigma_1^{(n)^2} & \sigma_1^{(n)}\sigma_2^{(n)} & \sigma_2^{(n)^2} & \sigma_6^{(n)^2} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_{11} \\ F_{12} \\ F_{22} \\ F_{66} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ - \\ \\ - \\ 1 \end{Bmatrix} \qquad (766)$$

$$[\sigma]\ \{F\} = \{1\} \tag{767}$$

$[\sigma]$: $6 \times n$ matrix

$\{F\}$: 1×6 matrix   n : no. of tests

$\{1\}$: $1 \times n$ matrix

$$[\sigma]^T [\sigma] \{F\} = [\sigma]^T \{1\} \tag{768}$$

$$\therefore \{F\} = ([\sigma]^T [\sigma])^{-1} [\sigma]^T \{1\} \tag{769}$$

Minimum no. of tests required = no. of components of F

Distribution of f

$$f_i(\sigma_i) = F_1\sigma_1^{(i)} + F_2\sigma_2^{(i)} + F_{11}\sigma_1^{(i)^2} + F_{12}\sigma_1^{(i)}\sigma_2^{(i)} + F_{22}\sigma_2^{(i)^2} + F_{66}\sigma_6^{(i)^2} \tag{770}$$

$$\bar{f} = \frac{1}{n}\sum_i f_i \tag{771}$$

$$s = \left[\frac{1}{n}\sum_i (f_i - \bar{f})^2\right]^{1/2} \tag{772}$$

$$F(f) = 1 - \exp\left[-(f/f_o)^{\alpha}\right] \tag{773}$$

If no coupling is assumed between σ_1 and (σ_2, σ_6), then one can define

$$f_f(\sigma_1) = F_1\sigma_1 + F_{11}\sigma_1^2 , \tag{774}$$

$$f_m(\sigma_2, \sigma_6) = F_2\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 . \tag{775}$$

(1) Sample problem

Failure occurred at the following values of σ_2 and σ_6.

Determine F_2, F_{22}, F_{66}, and f.

σ_2, MPa	9.99	12.78	25.03	11.05	-18.62	12.82
σ_6, MPa	37.30	-45.26	5.62	51.34	69.50	0

Solution

$$[\sigma]^T [\sigma] = \begin{bmatrix} 1423 & 14761 & -33868 \\ & 581209 & 2350972 \\ \text{symmetric} & & 34472975 \end{bmatrix}$$

$$[\sigma]^T \{1\} = \begin{Bmatrix} 43.1 \\ 1423 \\ 9545 \end{Bmatrix}$$

$F_2 = 3.035 \times 10^{-2}$ (MPa)$^{-1}$

$F_{22} = 6.035 \times 10^{-4}$ (MPa)$^{-2}$

$F_{66} = 2.655 \times 10^{-4}$ (MPa)$^{-2}$

f : 0.73 , 1.03, 1.15, 1.11, 0.93, 0.49

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SECTION XV

STRUCTURAL ELEMENTS

1. LAMINATED COMPOSITE BEAMS - STATIC BEHAVIOR

a. Formulation

The classical beam bending equation is derived by combining the appropriate moment-curvature relationship and equilibrium equation.

$$K_x = \frac{-d^2w}{dx^2} = \frac{M}{EI} \tag{776}$$

$$\frac{d^2M}{dx^2} + q(x) = 0 \tag{777}$$

Then

$$EI \frac{d^4w}{dx^4} = q(x) \tag{778}$$

For laminated beams it is convenient to replace EI by an "effective bending stiffness". The suitability of this approach has been investigated by deriving a beam theory from classical plate theory. The plate constitutive equations - inverted form

$$\begin{Bmatrix} e^o \\ \\ K \end{Bmatrix} = \begin{bmatrix} A' & B' \\ & \\ C' & D' \end{bmatrix} \begin{Bmatrix} N \\ \\ M \end{Bmatrix} \tag{779}$$

A solution for the deflection of a wide beam, simply supported, loaded by a uniform moment is available. The apparent bending stiffness is

$$(EI)_A = \frac{bR^2}{[D_{11}' R^2 + D_{16}' R + D_{12}]} \tag{780}$$

where R-length to width ratio.

The effect of bending anisotropy can be significant.

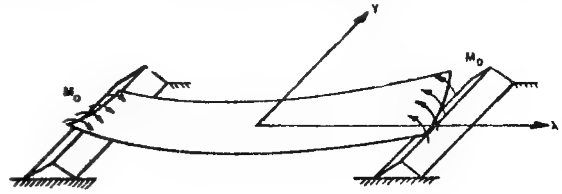
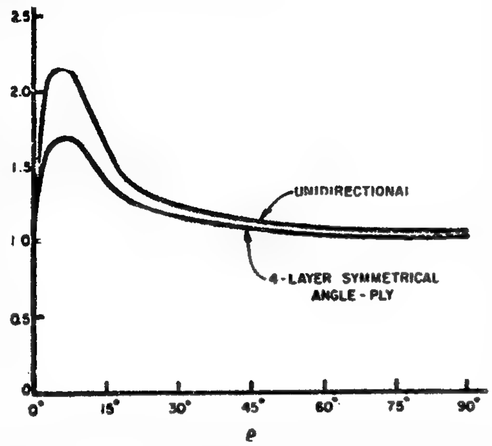


Figure 165 Unidirectional beam under pure bending. Fibers oriented θ to x axis. Twisting curvature "lifts" beam at supports.



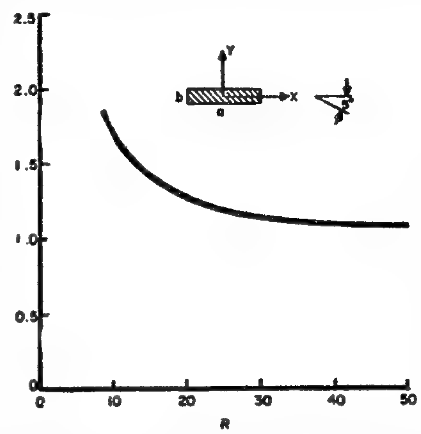


Figure 166 $(EI)_A/(b/D_{11}')$ as a function of θ; as a function of R for a 10° off axis unidirectional beam.

The D' matrix is defined as $[D-BA^{-1}B]^{-1}$ and involves the B-coupling matrix for unsymmetrical laminates.

Effects of bending anisotropy on $(EI)_A$ depend on E_{11}/E_{22}, h, R, and stacking sequence.

Effects of B-coupling on $(EI)_A$ depend on E_{11}/E_{22}, h and stacking sequence.

R large or D_{16}'/D_{11}' small

$$(EI)_A) \simeq \frac{b}{D_{11}'} \tag{781}$$

$D_{16}' = 0$

$$(EI)_A = \frac{b}{D_{11}'} \sim (D_{11} - B_{11}^2/A_{11})b \tag{782}$$

$(D_{11} - B_{11}^2/A_{11})b$ is a "reduced bending stiffness" and is comparable to the transformed area method of computing EI.

Mid-plane symmetry and no bending anisotropy, $B_{ij} = D_{16} = 0$

$$(EI)_A \sim D_{11}b \tag{783}$$

occasionally, one finds

$$(EI)_A = \sum_k E_x^k I^k \tag{784}$$

where I^k is computed about the beam mid-surface

$$\text{and} \qquad \frac{1}{E_x} = \frac{\cos^4\theta}{E_{11}} + \left(\frac{1}{G_{12}} + \frac{2\nu_{12}}{E_{22}}\right)\sin^2\theta\cos^2\theta + \frac{\sin^4\theta}{E_{22}} \tag{785}$$

Significant error can result if Eq. (784) is used when B coupling and bending anisotropy are present.

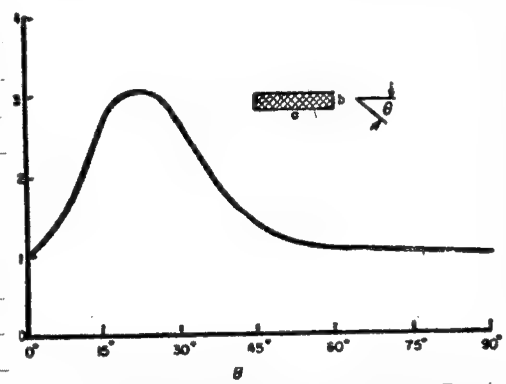


Figure 167 Ratio of bending stiffness computed by Eqs. (781) and (784-785)

The assumed linear strain in a beam results from considering bending deformation only. Shearing deformation can be significant in composite beams.

Example. Simply supported beam, point loaded at center, deflection computed for the center (by virtual work).

$$\delta_c = \frac{P\ell}{4EI}\Big[\underbrace{\frac{\ell^2}{12}}_{\text{bending term}} + \underbrace{\lambda n \Gamma^2}_{\text{shear term}}\Big] = \frac{P\ell}{4EI}[\delta_b + \delta_s] \tag{786}$$

where

ℓ = beam length $\qquad \Gamma^2 = I/A$

$n = E_x/G_{xz}$ $\qquad \lambda$ = form factor

consider a unidirectional composite, fibers oriented to x-axis

ℓ = 8, 4, 2 in

$E_{11} = E_x = 25 \times 10^6$ psi, $\quad G_{13} = G_{xz} = 0.5 \times 10^6$ psi

b = 1 in, t = 0.25 in. $\qquad I = 0.0013\ \text{in}^4$

d = 1.2 (rectangular beam) $\qquad A = 0.25\ \text{in}^2$

$\Gamma^2 = 0.0052\ \text{in}^2$

	$\delta_b = \ell^2/12$	$\delta_s = n\Gamma^2$	$\delta_b/(\delta_s+\delta_b)$	$\delta_s/(\delta_s+\delta_b)$
$\ell = 8$	5.333	0.312	0.945	0.055
$\ell = 4$	1.333	0.312	0.810	0.190
$\ell = 2$	0.333	0.312	0.516	0.484

Beam deflections (negligible or zero bending anisotropy)

$$(EI)_A = (D_{11} - B_{11}^2/A_{11})b \qquad (787)$$

and

$$(EI)_A\ W(x) = \iiiint q(x)\, dx \qquad (788)$$

as in classical beam theory

Beam stresses

normal $$\sigma_x^k = E_x^k(e_x^o + z^k\, k_x) \qquad (789)$$

$$e_x^o = B_{11}{}'\, M \qquad (790)$$

$$k_x = D_{11}{}'\, M \qquad M = b\, M_x \qquad (791)$$

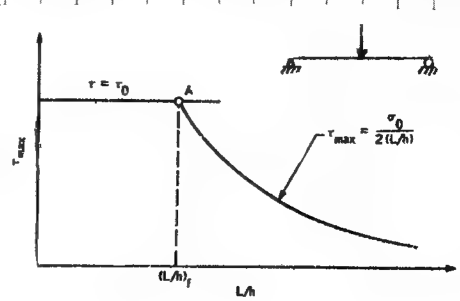
shear

(rect. cross-section) $$\tau_{xz} \simeq \frac{V}{2I}\left[\left(\frac{h}{2}\right)^2 - y_k^2\right] \text{ (str. of mat'ls)} \qquad (792)$$

y_k - distance from mid-surface to k^{th} interface

$$\tau_{xz}\,(\max) = \frac{3}{2}\ \frac{V}{A} \qquad (793)$$

Figure 168 First failure in a beam can be either shear or flexural stress depending on span-to-depth ratio, L/h.



b. Problems

(1) Assess the effect of B-coupling on $(EI)_A$ for

a) $0°_{24}/90°_{24}$

b) $0°_{12}/90°_{12}/0°_{12}/90°_{12}$

c) $(0°_2/90°_2)_{12}$

where $E_{11} = 20 \times 10^6$ psi $\qquad G_{12} = 0.5 \times 10^6$ psi

$E_{22} = 1 \times 10^6$ psi $\qquad \nu_{12} = 0.3$

$t = 0.005$ in.

(2) Use the equilibrium equations given below and Eq. (779) to derive a governing equation for the beam-column.

$$\frac{dN_x}{dx} = 0 \quad , \quad \frac{d^2M_x}{dx^2} + N_x \frac{d^2W}{dx^2} + q(x) = 0$$

2. LONG CYLINDRICAL TUBES (AXIAL LOAD, TORSION, PRESSURE, BEAM BENDING)

a. Formulation

Membrane theory with anisotropy and B-coupling included.

Stress fields more than one diameter from ends.

Define

$$a_{ij} = \frac{1}{h}\begin{bmatrix} A_{11} & A_{12} - B_{12}/R & A_{16} + B_{16}/R \\ A_{12} & A_{22} - B_{22}/R & A_{26} + B_{26}/R \\ A_{16} & A_{26} - B_{26}/R & A_{66} + B_{66}/R \end{bmatrix} \quad (794)$$

and

$$b_{ij} = a_{ij}^{-1} \quad (795)$$

now

$$e_i^o = \frac{b_{ij}}{h} N_j \quad (796)$$

and stress in a layer

$$\sigma_i = \bar{Q}_{ij} \frac{b_{jk}}{h} N_k = \frac{Z}{Rh}[\bar{Q}_{i6}\, b_{6j} - \bar{Q}_{i2}\, b_{2j}]\, N_j \quad (797)$$

N_x - axial load

$N_\theta = pR$ - pressure load

$N_{x\theta}$ - torsion load

Axial compliance

$$e_x^o = \frac{b_{11}}{h} N_x \quad ; \quad S_x = b_{11}/h \quad (798)$$

Effective axial modulus

$$E_x = 1/S_x = h/b_{11} \quad (799)$$

Cylindrical tube in beam bending (approx.)

$$E_x I \frac{d^4 w}{dx^4} = q(x) \quad (800)$$

Tubes with low A_{66} terms (all 0°, 0/90), include effect of shear deformation. Form factor, $\lambda = 2$

b. Problem

Design a laminated tube, $0^\circ_i/\pm45^\circ_j$ ply configuration, and calculate the stress in the innermost and outermost 45° and 0° layers.

$R_i = 2$ in.

$E_{11} = 20 \times 10^6$ psi $\quad e_1^{\ell} = +0.066, -0.005 \quad N_x = 2300$ lbs/in.

$E_{22} = 1 \times 10^6$ psi $\quad e_2^{\ell} = +0.003, -0.006 \quad N_{x\theta} = 500$ lbs/in.

$G_{12} = 0.5 \times 10^6$ psi $\quad e_6^{\ell} = \pm 0.015$

$\nu_{12} = 0.3$

$t = 0.005$ in.

3. COLUMNS (COMPRESSION LOADED 1-D MEMBERS)

a. Formulation

Laminated column $D_{16} = 0$ or D_{16}/D_{11} very small

Bending stiffness

$$(EI)_A = (D_{11} - B_{11}^2/A_{11})b \qquad (801)$$

Column equation

$$(EI)_A \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0 \qquad (802)$$

Euler buckling load, simply supported columns

$$P_e = \frac{\pi^2 (EI)_A}{L^2} \qquad (803)$$

For general end restraints $\frac{dw}{dx} = M_1/\alpha_1$ @ $x = o$

$$\frac{dw}{dx} = M_2/\alpha_2 \text{ @ } x = L \qquad (804)$$

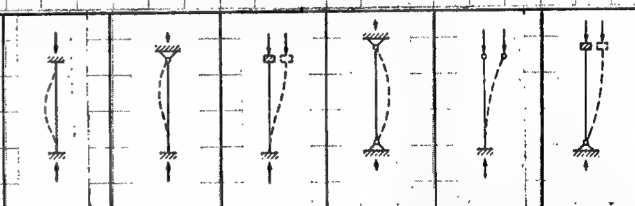
defining $\lambda_1 = \frac{(EI)_A}{\alpha_1 L}$, $\lambda_2 = \frac{(EI)_A}{\alpha_2 L}$

$$\phi = kL \quad , \quad k^2 = p/(EI)_A \qquad (805)$$

the characteristic equation is

$$(1-\lambda_1-\lambda_2-\lambda_1\lambda_2\phi^2)\,\phi \sin\phi + (2+\lambda_2\phi^2)\cos\phi - 2 = 0 \qquad (806)$$

For special boundary conditions, the effective length approach can be used with

$L_{eff} = KL$	Buckled shape of column is shown by dashed line						
	Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
$P_{cr} = \frac{\pi^2 (EI)_A}{L_{eff}^2}$	Recommended design value when ideal conditions are approximated	0.65	0.80	1.2	1.0	2.10	2.0
	End condition code	Rotation fixed and translation fixed Rotation free and translation fixed Rotation fixed and translation free Rotation free and translation free					

Role of B_{11} coupling

(1) Reduced bending stiffness - lower P_{cr}

(2) The reduced bending stiffness can also be calculated by the transformed area method. This illustrates the additional effect of shifting the columns effective centroid (neutral axis for bending)

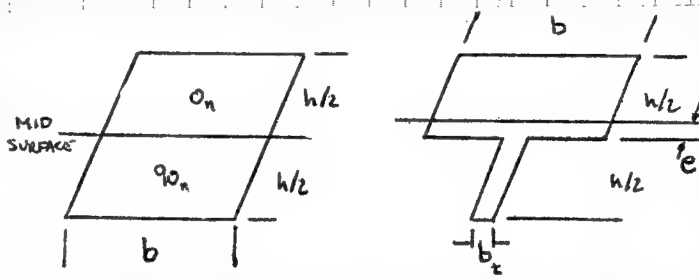


Figure 169 A 'transformed section', all 0°, $b_t=(E_{22}/E_{11})b$.

Therefore, axial loads applied coincident to the mid surface produce bending deformations with possible failure due to excessive deformation with $P < P_{cr}$ (reduced)

Role of low interlaminar shear modulus G_{xz}

Reduces resistance (total effective stiffness) to out-of-plane deformation (buckling)

$$\bar{P}_{cr} = k\,P_{cr} \tag{807}$$

where

$$k = \frac{1}{1 + \lambda \dfrac{P_{cr}}{AG}} \quad \text{- form factor} \tag{808}$$

Laminated circular tube - compression

no anisotropy

$$(EI)_A = E_x I \tag{809}$$

E_x defined by

Secondary column failure modes

For cylindrical tubes, high R/h, high compressive load, there can be cylindrical buckling - many analytical tools available.

Anisotropy and heterogeneous - Cheng and Ho

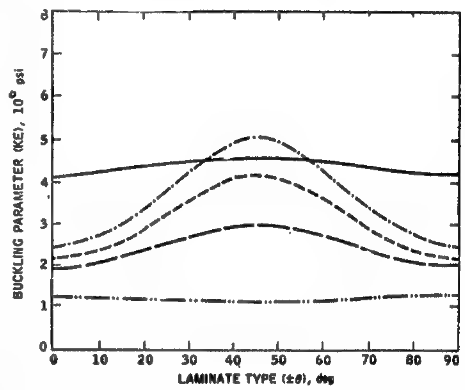
heterogeneous - Tsai

orthotropic - NASA and Air Force Design Guides, Handbooks

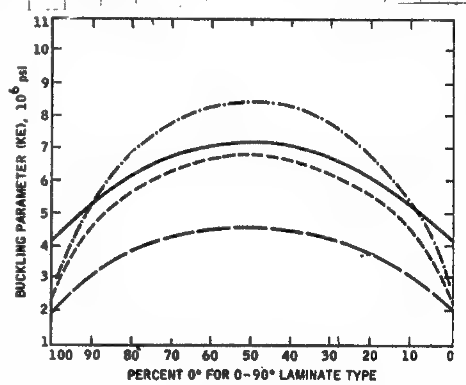
Homogeneous, orthotropic approximation

$$\sigma_{cr} = KE_1 (h/D) \tag{810}$$

K is evaluated for various laminates and materials



(a)



(b)

Legend (a):

- ——— Boron Epoxy
- —·— U. H. M. Graphite/Epoxy
- – – – H. M. Graphite/Epoxy
- —— —— H. S. Graphite/Epoxy
- —··— Fiberglas

Legend (b):

- ——— Boron/Epoxy
- —·— U. H. M. Graphite/Epoxy
- – – – H. M. Graphite/Epoxy
- —— —— H. S. Graphite/Epoxy

Figure 170 Buckling parameter:
(a) [± θ] laminate;
(b) [0/90] laminate.

b. Problems

(1) Develop an expression for the layer stresses in a laminated beam-column.

(2) Size a laminated tube for the given compressive load and geometry

$P = 20{,}000$ lbs $\quad R = 2$ in. $\quad L = 15$ ft.

$K = 1$

$E_{11} = 40 \times 10^6$ psi $\quad G_{12} = 0.5 \times 10^6$ psi $\quad t = 0.005$ in.

$E_{22} = 1 \times 10^6$ psi $\quad \nu_{12} = 0.3$

(3) Assess the effect of low shear stiffness on $\bar{P}_{cr}$.

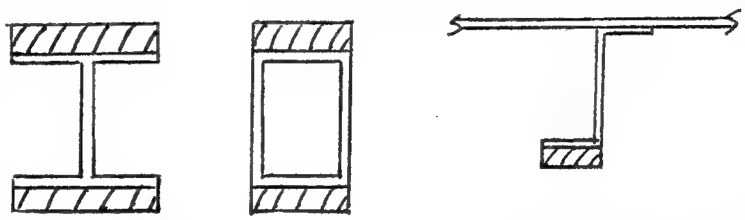
(4) Re-evaluate the design of (2) for local instability.

4. COMPOSITES FOR SELECTIVE REINFORCEMENT OF STRUCTURAL ELEMENTS

a. Formulation

Standard structural forms can be reinforced by unidirectional (0°) composites for enhanced specific stiffness and strength.

Examples



Beams   Panel Stringers

Mid plane symmetric

$$(EI)_A = \Sigma (EI) \qquad (811)$$

no symmetry

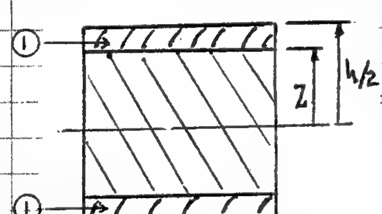
$$(EI)_A = D_x - B_x^2 / A_x \qquad (812)$$

or transformed area method

Specific stiffness

$$(EI)_A / \rho_T = \frac{\Sigma (EI)}{\frac{\Sigma(\rho V)}{V}} = \Sigma\left(\frac{EI}{\rho V}\right) \qquad (813)$$

Shear stress in adhesive



$$\tau = \frac{VQ}{\bar{I}b} \qquad (814)$$

$\bar{I}$ - moment of inertia, transformed area, all material ①

b - width

V - shear force at x

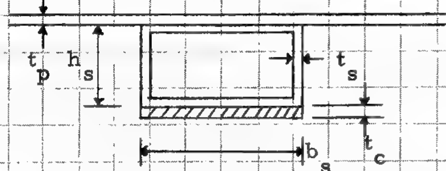
Q - first moment of area, composite reinforcement about centroid axis

Secondary failure mode

local buckling - compare to sandwich beam face sheet.

b. Problem

An aluminum stringer reinforced panel is to be upgraded for higher specific stiffness. Unidirectional boron/epoxy strip reinforcement is chosen. Evaluate the change in specific stiffness $(EI)_A$



$t_p = 0.050$ in. $t_s = 0.010$ in.

$h_s = 0.50$ in. $t_c = 0.0$

$b_s = 1.0$ in $b = 4.0$ in.

$\bar{A}_x = A_{11} + E_s A_s / b$

$\bar{B}_x = B_{11} + Z_s E_s A_s / b$

$\bar{D}_x = D_{11} + E_s I_s / b + Z_s^2 E_s A_s / b$

b - stringer spacing

subscript s - relates to stringer

$E_{11} = 35 \times 10^6$ psi $G_{12} = 1 \times 10^6$ psi

$E_{22} = 3.5 \times 10^6$ $\nu_{12} = 0.25$

$\rho_{AL} = 0.1$ lbs/in^3 $\rho_c = 0.075$ lbs/in^3

5. THERMAL STRESSES

Tubes

$$\sigma_i = \overline{Q}_{ij} \left[\frac{b_{jk}}{h} \overline{N}_k - \overline{e}_j \right] + \frac{Z}{Rh} \left[\overline{Q}_{i6} b_{6j} - \overline{Q}_{i2} b_{2j} \right] \overline{N}_j \tag{815}$$

$$\text{where } \overline{N}_i = \int_{-h/a}^{h/2} \overline{Q}_{ij} \, \overline{e}_j \, dz \tag{816}$$

$$\overline{e}_i = \overline{\alpha}_i \, \Delta T \tag{817}$$

$\overline{\alpha}_i$ = lamina coef. of thermal expansion in x, θ coord.

$(\overline{\alpha}_x, \overline{\alpha}_\theta, \overline{\alpha}_{x\theta}$ from $\alpha_1, \alpha_2)$

6. SIZING FOR STIFFNESS

a. Formulation

Stiffness critical designs : bending - deflection

compression - stability

dynamic - frequency control

In 1-D structures (beams, columns), unidirectional laminates will predominate in designs.

In 2-D structures (plates, shells), angle-plied laminates become more efficient.

Plate stability (assume quasi-homogeneous, orthotropic : $B_{ij} = 0$, D_{16}, D_{26} - negligible) simply supported, compression loaded on X edges

$$N_{xcr} = \frac{2\pi^2}{b^2}\left\{(D_{11}D_{22})^{1/2} + D_{12} 2D_{66}\right\} \tag{818}$$

or

$$\sigma_{xcr} = K(h/b)^2 \tag{819}$$

Numerous sources exist which give K for different materials and laminate configurations. Examples are given.

LDP 266

$\pm\theta$ angle ply, symmetric, laminates

— boron/epoxy

- - - U.H.M. graphite/epoxy

—·— H.M. " "

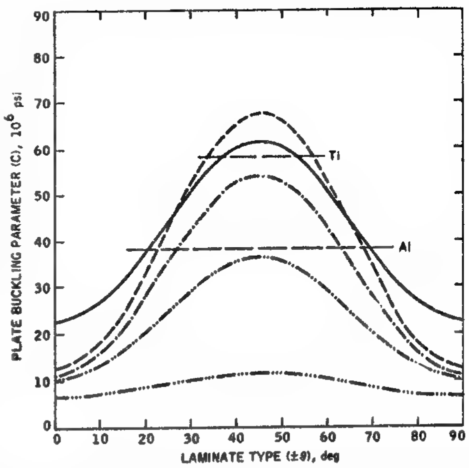
—··— H.S. " "

—···— E-glass/epoxy

example

KE

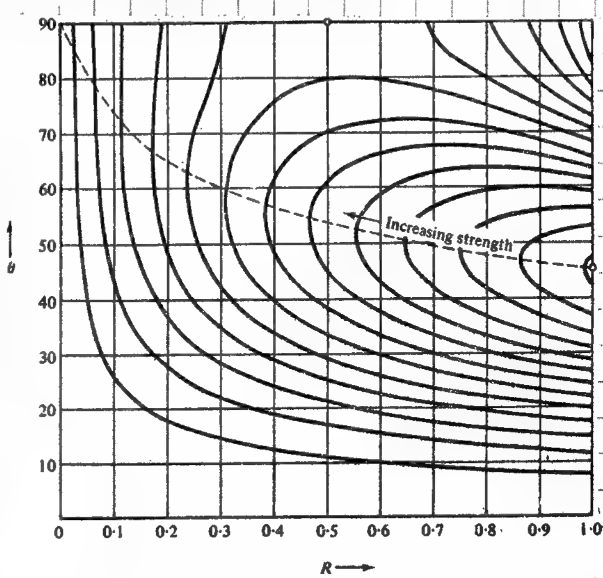
(x 10^6 psi)



$\pm\theta_n/0_M$ angle ply with 0^o plies

R - ratio of $\pm\theta$ plies to total number of plies

example

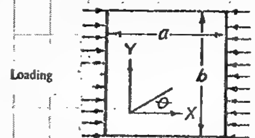


K

Figure 171 Example calculations.

$0^\circ/90^\circ/\pm45^\circ$ laminates example

PLATE STABILITY OF 0°, 90°, AND $\pm45^\circ$ FAMILY OF PLATES



$$\sigma_{cr} = K_{cr} \left(\frac{t}{b}\right)^2$$

t - Laminate Thickness

Thickness Fraction of Ply Orientation, %				$K_{cr} \times 10^{-6}$ psi		
45°	0°	90°	135°	Quasi-Homogeneous Plying Order	Symmetric Plying Order	Unbalanced Plying Order
50	0	0	50	54.5	45.8	27.1
40	20	0	40	50.0	42.7	25.2
30	40	0	30	45.5	40.7	25.8
20	60	0	20	41.1	39.1	29.0
10	80	0	10	36.5	36.8	33.5
0	100	0	0	32.0	32.0	32.0
40	0	20	40	50.0	42.8	25.2
30	20	20	30	45.5	40.8	27.7
20	40	20	20	41.1	39.1	29.6
10	60	20	10	36.5	36.8	30.4
0	80	20	0	32.0	32.0	24.3
30	0	40	30	45.5	40.5	25.8
20	20	40	20	41.1	39.2	29.6
10	40	40	10	36.5	36.8	28.2
0	60	40	0	32.0	32.0	18.5
20	0	60	20	41.1	38.9	29.0
10	20	60	10	36.5	36.8	30.4
0	40	60	0	32.0	32.0	18.5
10	0	80	10	31.2	32.6	33.2
0	20	80	0	32.0	32.0	24.3
0	0	100	0	20.8	20.8	20.8
Quasi-Isotropic						
25	25	25	25	43.3	40.0	28.6

1. Quasi-homogeneous plying orders contain a sufficient number of consistently repeated sequences of lamina to make negligible the effects of resulting bending couples. The basis for these sequences is 45°, 0°, 90°, and 135°.
2. Symmetric plying orders contain balanced ply orientations where the sequence of lamina directions are exactly mirror image about the laminate midplane. The bases for these are 45°, 0°, 90°, 135°, 90°, 0°, and 45°.
3. Unbalanced plying orders contain only one sequence of orientations based on 45°, 0°, 90°, and 135°.

It is evident that for Nx loading the $\pm 45^{o}$ plate gives the maximum resistance to buckling. This results from the dominant $2D_{66}$ term in Eq. (818). Since strength must also be adequate, 0^{o} plies may be added or $\pm\theta$ laminates at lower angles may be considered.

Hole of stacking sequence - example

$\pm 45^{o}$ plies on the outer surfaces produces
the maximum D_{66}.

TABLE 73 EFFECTS OF ORIENTATION SEQUENCES ON STABILITY OF SYMMETRICAL (BALANCED) QUASI-ISOTROPIC PLATES

Orientation Sequence of Layers								$K_{cr} \times 10^{-6}$ psi
45°	135°	0°	90°	90°	0°	135°	45°	50.6
135°	0°	45°	90°	90°	45°	0°	135°	44.0
0°	135°	45°	90°	90°	45°	135°	0°	40.6
45°	0°	90°	135°	135°	90°	0°	45°	40.0
0°	135°	90°	45°	45°	90°	135°	0°	37.7
0°	90°	45°	135°	135°	45°	90°	0°	34.7

1. All plates are equal in thickness.
2. Loading and laminate plane coordinates are the same as in Table 72

Consider compression strength

$$\sigma_x = N_x/h \tag{820}$$

Requiring $\sigma_x = \sigma_{xcr}$ and defining as $\sigma_{optimum}$

$$\sigma^3_{OPT} = \sigma_{xcr} \cdot \sigma^2_x = K(h/b)^2 (N_x/h)^2 \tag{821}$$

eliminates h as

$$\sigma_{OPT} = K^{1/3}\left(\frac{N_x}{b}\right)^{2/3} \tag{822}$$

$\left(\frac{N_x}{b}\right)^{2/3}$ - structural index

Combining with a laminate failure criterion, the optimum laminate can be designed for the specific class being considered ($\pm\theta$, $\pm\theta_m/0^o_n$, $0^o/90^o/\pm45^o$)

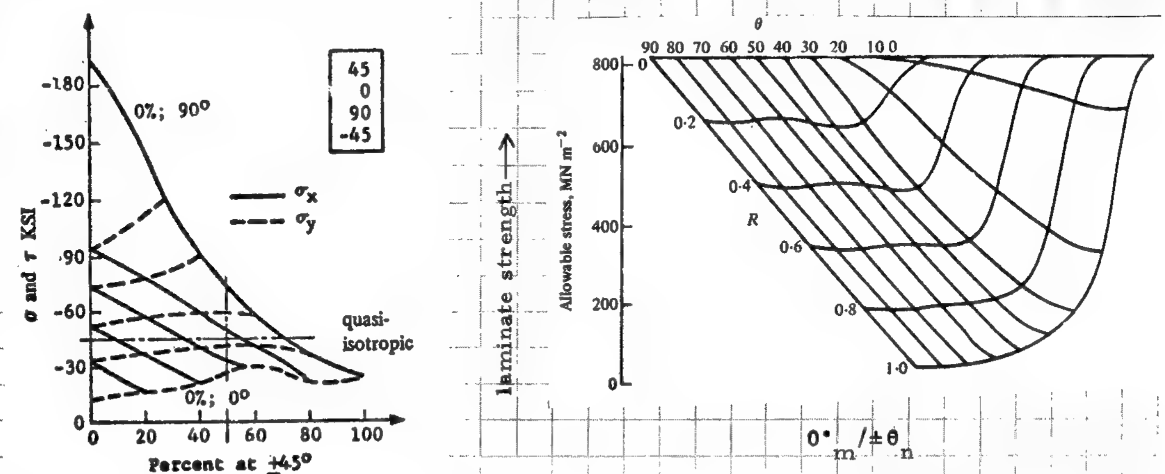


Figure 172 Examples of Uniaxial strength allowables (compression).

b. Problems

(1) Calculate the individual bending stiffnesses (D's) of Eq. (818) for the following laminates and material

$[+45, -45, +45, -45]_s$

$[0, 90, 0, 90]_s$

$[+45, -45, 0, 90]_s$

$E_{11} = 30 \times 10^6$ psi $\quad G_{12} = 0.5 \times 10^6$ psi $\quad t = 0.005$

$E_{22} = 1.5 \times 10^6$ psi $\quad \nu_{12} = 0.3$

(2) Using the charts and tables available for strength and K_{cr} for $(0^o/\pm45^o/90^o)$ laminate class, determine a minimum weight design for a compression panel where

$N_x = 5{,}000$ lbs/in. $\quad N_y = N_{xy} = 0$

$a = b = 10$ inches

Plot WT/b vs N_x/b for stability controlled and strength controlled designs

Plot σ_{OPT} vs $(N_x/b)^{2/3}$

7. DESIGN PROBLEM - STRINGER REINFORCED COMPRESSION PANEL

Discussion and notes

SECTION XVI

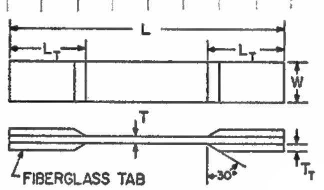
SPECIMEN CONFIGURATIONS AND LOADINGS

1. TESTS FOR LAMINA PROPERTIES

a. Properties to Be Measured

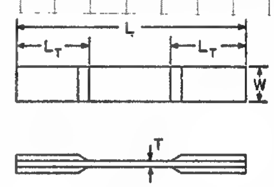
1. TENSION - E_L, E_T, V_{LT}, F_L^t, F_T^t
   COUPON, SANDWICH BEAM, TUBE
2. COMPRESSION - E_L, E_T, V_{LT}, F_L^c, F_T^c
   COUPON, SANDWICH BEAM, TUBE, SHORT BAR
3. INPLANE SHEAR - G_{LT}, F_{LT}^s
   [±45] COUPON, CROSS SANDWICH BEAM, TUBE
4. FLEXURE - E^f, F^f
   BEAM
5. INTERLAMINAR SHEAR - G_{IS}, F_{IS}
   SHORT BEAM
6. GENERAL STATE OF STRESS
   OFF-AXIS COUPON, TUBE, CROSS SANDWICH BEAM

b. Tension and Compression



ORIENTATION	L	W	T	L_T	T_T
0°	9	0.5	6 pl.	1.5	1/16
90°	9	1	8 pl.	1.5	1/16
±45°	9	1	8 pl.	1.5	1/16

(a)



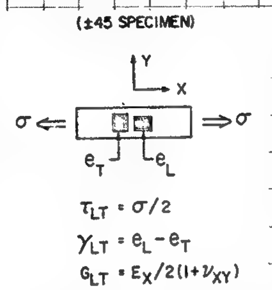
	CELANESE	SWRI*
L	5.5	8.5
W	.25	.50
T	—	—
L_T	2.5	2.5

*SIDE SUPPORT

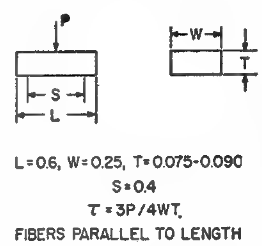
(b)

Figure 173. Specimen configurations: (a) tension; (b) compression.

c. Longitudinal and Interlaminar Shear

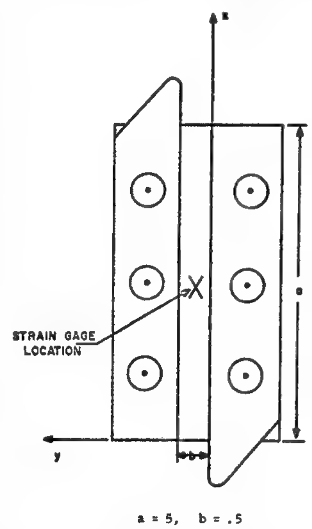


(a)

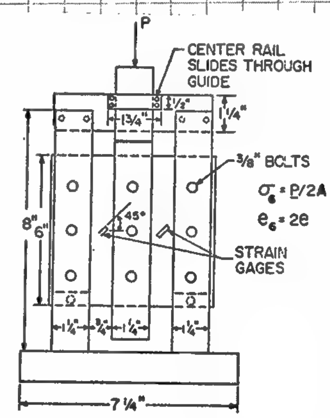


(b)

Figure 174 Specimen configurations: (a) longitudinal shear; (b) interlaminar shear.



(a)



(b)

Figure 175 Rail shear tests: (a) single; (b) double.

d. Flexure Tests

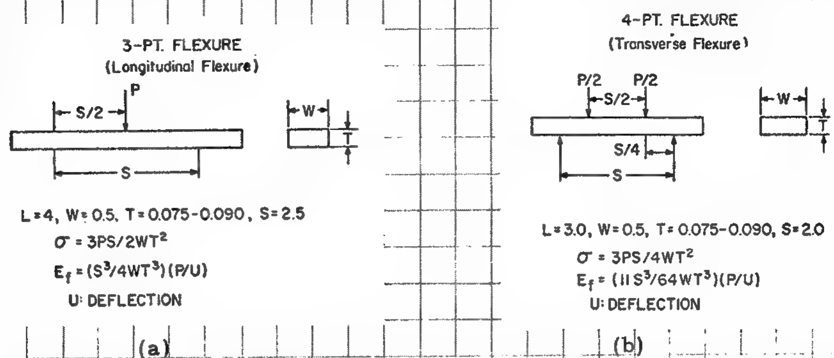


Figure 176 Flexure test configurations: (a) 3-pt.; (b) 4-pt.

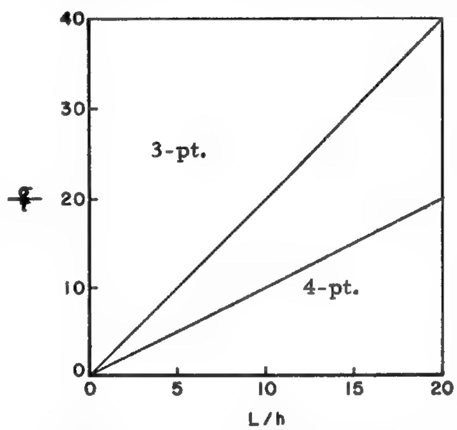
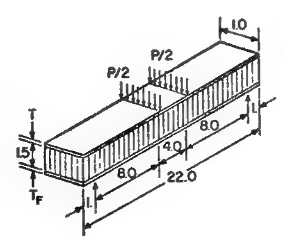


Figure 177 Relation between maximum tensile stress and maximum shear stress in flexure tests.

e. Sandwich Beam Test



SANDWICH BEAM DETAILS

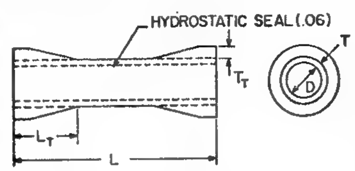
ORIENTATION	No. OF PLIES	SECOND FACE MATERIAL	CORE ALUMINUM HONEYCOMB DENSITY (pcf)
0°	6	STAINLESS STEEL (.125" THICK)	23
90°	8	GLASS/EPOXY (.08" THICK)	45

$$\sigma = 4P/T\left[1.5 + (T+T_F)/2\right]$$

Figure 178 Sandwich beam configuration.

UDF 266

f. Tube Test

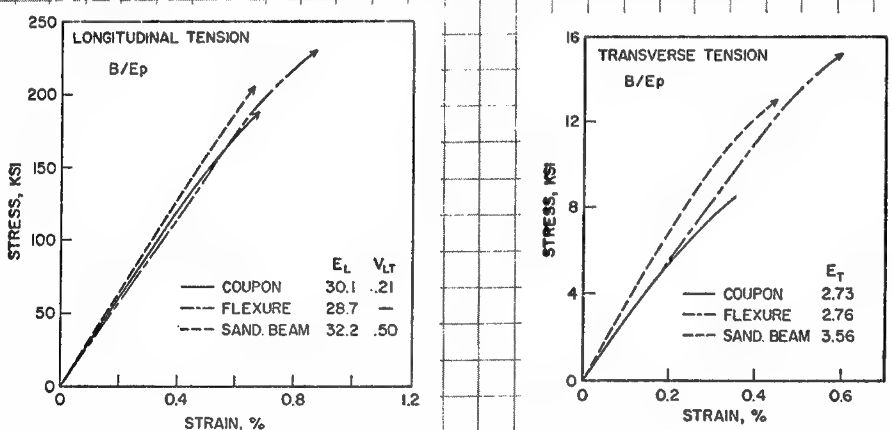


L	D	T	L_T	T_T
13	4	.04–.06	4–4.5	.7
12	2	.05–.06	1, 2	(1)
10	3	.04–.06	3	.5(2)
10	1	.04–.08	3	.75

(1) INSIDE SUPPORT
(2) INSIDE TAB

Figure 179 Tube configuration.

g. Comparison Among Different Test Methods



(a) (b)

Figure 180 Tensile properties: (a) longitudinal; (b) transverse. [43]

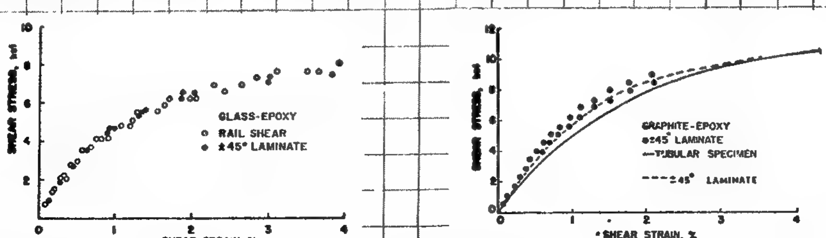


Figure 181 Shear stress-strain relations. [44]

TABLE 74 COMPRESSION PROPERTIES [45]

ORIENTATION	TYPE OF SPECIMEN	TEST TEMP. (^{o}F)	E (MSI)	V	σ_{ult} (KSI)	ϵ_{ult} (%)
0^{o}	COUPON	RT	23	.34	218	.95
	S. BEAM	RT	23	.39	247	1.42
	COUPON	350	22.5	.31	206	1.16
	S. BEAM	350	21.4	.50	214	1.30
90^{o}	COUPON	RT	1.64	.01	36.3	2.50
	S. BEAM	RT	1.76	.02	35.7	2.36
	COUPON	350	1.60	.01	30.4	2.29
	S. BEAM	350	1.76	.03	28.6	2.17

T300/5208

l_G = 0.5" FOR COUPON

2. PROBLEMS RESULTING FROM ANISOTROPY AND NONHOMOGENEITY

a. Coupling Effects

Shear coupling - S_{16}, $S_{26} \neq 0$

Extension - twisting coupling - B_{16}, $B_{26} \neq 0$

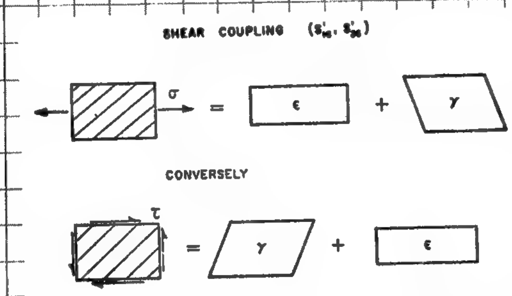


Figure 182 Effect of shear coupling.

b. Free Edge Effects

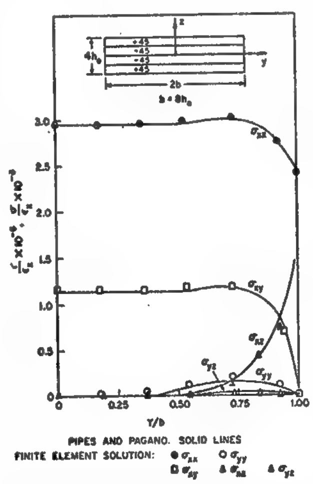


Figure 183 Stress concentration at free edge of $[45/-45]_s$ angle ply. [46]

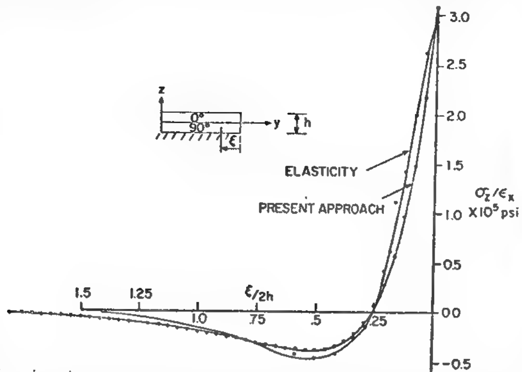


Figure 184 Interlaminar normal stress concentration. [47]

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APPENDIX A

SYSTEME INTERNATIONALE

1. METRIC PREFIXES

Prefix	Abbreviation	Multiplier
tera-	T	10^{12}
giga-	G	10^{9}
mega-	M	10^{6}
kilo-	k	10^{3}
hecto-	h	10^{2}
deca-	da	10
deci-	d	10^{-1}
centi-	c	10^{-2}
milli-	m	10^{-3}
micro-	μ	10^{-6}
nano-	n	10^{-9}
pico-	p	10^{-12}
femto-	f	10^{-15}
atto-	a	10^{-18}

2. CONVERSION EQUATIONS

1 kg = 2.20 lb	1 lb = .454 kg
1 kgm^{-3} = 3.61×10^{-5} lb/in^3	1 lb/in^3 = 2.77×10^{4} kgm^{-3}
1 N = .225 lb-f	1 lb-f = 4.45 N
1 N = .102 kg-f	1 kg-f = 9.81 N
1 Pa = 1.45×10^{-4} psi	1 psi = 6.895 Pa
1 MPa = .145 ksi	1 ksi = 6.895 MPa
1 GPa = .145 10^6 psi	10^6 psi = 6.895 GPa
1 Pa = .102 kg-f m^{-2}	1 kgf m^{-2} = 9.81 Pa
1 MPa = .102 kg-f mm^{-2}	1 kg-f mm^{-2} = 9.81 MPa
1 Nm^{-1} = .00571 lbf/in	1 lbf/in = 175 Nm^{-1}
1 Nm^{-1} = .102 kgf/m	1 kgf/m = 9.81 Nm^{-1}
1 Nm = 8.98 in-lbf	1 inlbf = $\frac{1}{8.98}$ Nm

APPENDIX B

TRIGONOMETRIC FUNCTIONS

THETA (DEG)	COS 2θ	COS 4θ	SIN 2θ	SIN 4θ	THETA (DEG)
-90	-1.00000	1.00000	-.00000	.00000	-90
-85	-.98481	.93969	-.17365	.34202	-85
-80	-.93969	.76604	-.34202	.64279	-80
-75	-.86603	.50000	-.50000	.86603	-75
-70	-.76604	.17365	-.64279	.98481	-70
-65	-.64279	-.17365	-.76604	.98481	-65
-60	-.50000	-.50000	-.86603	.86603	-60
-55	-.34202	-.76604	-.93969	.64279	-55
-50	-.17365	-.93969	-.98481	.34202	-50
-45	.00000	-1.00000	-1.00000	-.00000	-45
-40	.17365	-.93969	-.98481	-.34202	-40
-35	.34202	-.76604	-.93969	-.64279	-35
-30	.50000	-.50000	-.86603	-.86603	-30
-25	.64279	-.17365	-.76604	-.98481	-25
-20	.76604	.17365	-.64279	-.98481	-20
-15	.86603	.50000	-.50000	-.86603	-15
-10	.93969	.76604	-.34202	-.64279	-10
-5	.98481	.93969	-.17365	-.34202	-5
0	1.00000	1.00000	.00000	.00000	0
5	.98481	.93969	.17365	.34202	5
10	.93969	.76604	.34202	.64279	10
15	.86603	.50000	.50000	.86603	15
20	.76604	.17365	.64279	.98481	20
25	.64279	-.17365	.76604	.98481	25
30	.50000	-.50000	.86603	.86603	30
35	.34202	-.76604	.93969	.64279	35
40	.17365	-.93969	.98481	.34202	40
45	.00000	-1.00000	1.00000	.00000	45
50	-.17365	-.93969	.98481	-.34202	50
55	-.34202	-.76604	.93969	-.64279	55
60	-.50000	-.50000	.86603	-.86603	60
65	-.64279	-.17365	.76604	-.98481	65
70	-.76604	.17365	.64279	-.98481	70
75	-.86603	.50000	.50000	-.86603	75
80	-.93969	.76604	.34202	-.64279	80
85	-.98481	.93969	.17365	-.34202	85
90	-1.00000	1.00000	.00000	-.00000	90

APPENDIX C

NORMAL DISTRIBUTION

z	$F(z)$	$1 - F(z)$	z	$F(z)$	$1 - F(z)$	z	$F(z)$	$1 - F(z)$	z	$F(z)$	$1 - F(z)$
.00	.5000	.5000	.50	.6915	.3085	1.00	.8413	.1587	1.50	.9332	.0668
.01	.5040	.4960	.51	.6950	.3050	1.01	.8438	.1562	1.51	.9345	.0655
.02	.5080	.4920	.52	.6985	.3015	1.02	.8461	.1539	1.52	.9357	.0643
.03	.5120	.4880	.53	.7019	.2981	1.03	.8485	.1515	1.53	.9370	.0630
.04	.5160	.4840	.54	.7054	.2946	1.04	.8508	.1492	1.54	.9382	.0618
.05	.5199	.4801	.55	.7088	.2912	1.05	.8531	.1469	1.55	.9394	.0606
.06	.5239	.4761	.56	.7123	.2877	1.06	.8554	.1446	1.56	.9406	.0594
.07	.5279	.4721	.57	.7157	.2843	1.07	.8577	.1423	1.57	.9418	.0582
.08	.5319	.4681	.58	.7190	2810	1.08	.8599	.1401	1.58	.9429	.0571
.09	.5359	.4641	.59	.7224	.2776	1.09	.8621	.1379	1.59	.9441	.0559
.10	.5398	.4602	.60	.7257	.2743	1.10	.8643	.1357	1.60	.9452	.0548
.11	.5438	.4562	.61	.7291	.2709	1.11	.8665	.1335	1.61	.9463	.0537
.12	.5478	.4522	.62	.7324	.2676	1.12	.8686	.1314	1.62	.9474	.0526
.13	.5517	.4483	.63	.7357	.2643	1.13	.8708	.1292	1.63	.9484	.0516
.14	.5557	.4443	.64	.7389	.2611	1.14	.8729	.1271	1.64	.9495	.0505
.15	.5596	.4404	.65	.7422	.2578	1.15	.8749	.1251	1.65	.9505	.0495
.16	.5636	.4364	.66	.7454	.2546	1.16	.8770	.1230	1.66	.9515	.0485
.17	.5675	.4325	.67	.7486	.2514	1.17	.8790	.1210	1.67	.9525	.0475
.18	.5714	.4286	.68	.7517	.2483	1.18	.8810	.1190	1.68	.9535	.0465
.19	.5753	.4247	.69	.7549	.2451	1.19	.8830	.1170	1.69	.9545	.0455
.20	.5793	.4207	.70	.7580	.2420	1.20	.8849	.1151	1.70	.9554	.0446
.21	.5832	.4168	.71	.7611	2389	1.21	.8869	.1131	1.71	.9564	.0436
.22	.5871	.4129	.72	.7642	.2358	1.22	.8888	.1112	1.72	.9573	.0427
.23	.5910	.4090	.73	.7673	.2327	1.23	.8907	.1093	1.73	.9582	.0418
.24	.5948	.4052	.74	.7704	.2296	1.24	.8925	.1075	1.74	.9591	.0409
.25	.5987	.4013	.75	.7734	.2266	1.25	.8944	.1056	1.75	.9599	.0401
.26	.6026	.3974	.76	.7764	.2236	1.26	.8962	.1038	1.76	.9608	.0392
.27	.6064	.3936	.77	.7794	.2206	1.27	.8980	.1020	1.77	.9616	.0384
.28	.6103	.3897	.78	.7823	.2177	1.28	.8997	.1003	1.78	.9625	.0375
.29	.6141	.3859	.79	.7852	.2148	1.29	.9015	.0985	1.79	.9633	.0367
.30	.6179	.3821	.80	.7881	.2119	1.30	.9032	.0968	1.80	.9641	.0359
.31	.6217	.3783	.81	.7910	.2090	1.31	.9049	.0951	1.81	.9649	.0351
.32	.6255	.3745	.82	.7939	.2061	1.32	.9066	.0934	1.82	.9656	.0344
.33	.6293	.3707	.83	.7967	.2033	1.33	.9082	.0918	1.83	.9664	.0336
.34	.6331	.3669	.84	.7995	.2005	1.34	.9099	.0901	1.84	.9671	.0329
.35	.6368	.3632	.85	.8023	.1977	1.35	.9115	.0885	1.85	.9678	.0322
.36	.6406	.3594	.86	.8051	.1949	1.36	.9131	.0869	1.86	.9686	.0314
.37	.6443	.3557	.87	.8079	.1921	1.37	.9147	.0853	1.87	.9693	.0307
.38	.6480	.3520	.88	.8106	.1894	1.38	.9162	.0838	1.88	.9699	.0301
.39	.6517	.3483	.89	.8133	.1867	1.39	.9177	.0823	1.89	.9706	.0294
.40	.6554	.3446	.90	.8159	.1841	1.40	.9192	.0808	1.90	.9713	.0287
.41	.6591	.3409	.91	.8186	.1814	1.41	.9207	.0793	1.91	.9719	.0281
.42	.6628	.3372	.92	.8212	.1788	1.42	.9222	.0778	1.92	.9726	.0274
.43	.6664	.3336	.93	.8238	.1762	1.43	.9236	.0764	1.93	.9732	.0268
.44	.6700	.3300	.94	.8264	.1736	1.44	.9251	.0749	1.94	.9738	.0262
.45	.6736	.3264	.95	.8289	.1711	1.45	.9265	.0735	1.95	.9744	.0256
.46	.6772	.3228	.96	.8315	.1685	1.46	.9279	.0721	1.96	.9750	.0250
.47	.6808	.3192	.97	.8340	.1660	1.47	.9292	.0708	1.97	.9756	.0244
.48	.6844	.3156	.98	.8365	.1635	1.48	.9306	.0694	1.98	.9761	.0239
.49	.6879	.3121	.99	.8389	.1611	1.49	.9319	.0681	1.99	.9767	.0233
.50	.6915	.3085	1.00	.8413	.1587	1.50	.9332	.0668	2.00	.9773	.0227

UDP-266

x	$F(x)$	$1-F(x)$	x	$F(x)$	$1-F(x)$	x	$F(x)$	$1-F(x)$	x	$F(x)$	$1-F(x)$
2.00	.9773	.0227	2.50	.9938	.0062	3.00	.9987	.0013	3.50	.9998	.0002
2.01	.9778	.0222	2.51	.9940	.0060	3.01	.9987	.0013	3.51	.9998	.0002
2.02	.9783	.0217	2.52	.9941	.0059	3.02	.9987	.0013	3.52	.9998	.0002
2.03	.9788	.0212	2.53	.9943	.0057	3.03	.9988	.0012	3.53	.9998	.0002
2.04	.9793	.0207	2.54	.9945	.0055	3.04	.9988	.0012	3.54	.9998	.0002
2.05	.9798	.0202	2.55	.9946	.0054	3.05	.9989	.0011	3.55	.9998	.0002
2.06	.9803	.0197	2.56	.9948	.0052	3.06	.9989	.0011	3.56	.9998	.0002
2.07	.9808	.0192	2.57	.9949	.0051	3.07	.9989	.0011	3.57	.9998	.0002
2.08	.9812	.0188	2.58	.9951	.0049	3.08	.9990	.0010	3.58	.9998	.0002
2.09	.9817	.0183	2.59	.9952	.0048	3.09	.9990	.0010	3.59	.9998	.0002
2.10	.9821	.0179	2.60	.9953	.0047	3.10	.9990	.0010	3.60	.9998	.0002
2.11	.9826	.0174	2.61	.9955	.0045	3.11	.9991	.0009	3.61	.9998	.0002
2.12	.9830	.0170	2.62	.9956	.0044	3.12	.9991	.0009	3.62	.9999	.0001
2.13	.9834	.0166	2.63	.9957	.0043	3.13	.9991	.0009	3.63	.9999	.0001
2.14	.9838	.0162	2.64	.9959	.0041	3.14	.9992	.0008	3.64	.9999	.0001
2.15	.9842	.0158	2.65	.9960	.0040	3.15	.9992	.0008	3.65	.9999	.0001
2.16	.9846	.0154	2.66	.9961	.0039	3.16	.9992	.0008	3.66	.9999	.0001
2.17	.9850	.0150	2.67	.9962	.0038	3.17	.9992	.0008	3.67	.9999	.0001
2.18	.9854	.0146	2.68	.9963	.0037	3.18	.9993	.0007	3.68	.9999	.0001
2.19	.9857	.0143	2.69	.9964	.0036	3.19	.9993	.0007	3.69	.9999	.0001
2.20	.9861	.0139	2.70	.9965	.0035	3.20	.9993	.0007	3.70	.9999	.0001
2.21	.9864	.0136	2.71	.9966	.0034	3.21	.9993	.0007	3.71	.9999	.0001
2.22	.9868	.0132	2.72	.9967	.0033	3.22	.9994	.0006	3.72	.9999	.0001
2.23	.9871	.0129	2.73	.9968	.0032	3.23	.9994	.0006	3.73	.9999	.0001
2.24	.9875	.0125	2.74	.9969	.0031	3.24	.9994	.0006	3.74	.9999	.0001
2.25	.9878	.0122	2.75	.9970	.0030	3.25	.9994	.0006	3.75	.9999	.0001
2.26	.9881	.0119	2.76	.9971	.0029	3.26	.9994	.0006	3.76	.9999	.0001
2.27	.9884	.0116	2.77	.9972	.0028	3.27	.9995	.0005	3.77	.9999	.0001
2.28	.9887	.0113	2.78	.9973	.0027	3.28	.9995	.0005	3.78	.9999	.0001
2.29	.9890	.0110	2.79	.9974	.0026	3.29	.9995	.0005	3.79	.9999	.0001
2.30	.9893	.0107	2.80	.9974	.0026	3.30	.9995	.0005	3.80	.9999	.0001
2.31	.9896	.0104	2.81	.9975	.0025	3.31	.9995	.0005	3.81	.9999	.0001
2.32	.9898	.0102	2.82	.9976	.0024	3.32	.9996	.0004	3.82	.9999	.0001
2.33	.9901	.0099	2.83	.9977	.0023	3.33	.9996	.0004	3.83	.9999	.0001
2.34	.9904	.0096	2.84	.9977	.0023	3.34	.9996	.0004	3.84	.9999	.0001
2.35	.9906	.0094	2.85	.9978	.0022	3.35	.9996	.0004	3.85	.9999	.0001
2.36	.9909	.0091	2.86	.9979	.0021	3.36	.9996	.0004	3.86	.9999	.0001
2.37	.9911	.0089	2.87	.9979	.0021	3.37	.9996	.0004	3.87	.9999	.0001
2.38	.9913	.0087	2.88	.9980	.0020	3.38	.9996	.0004	3.88	.9999	.0001
2.39	.9916	.0034	2.89	.9981	.0019	3.39	.9997	.0003	3.89	1.0000	.0000
2.40	.9918	.0082	2.90	.9981	.0019	3.40	.9997	.0003	3.90	1.0000	.0000
2.41	.9920	.0080	2.91	.9982	.0018	3.41	.9997	.0003	3.91	1.0000	.0000
2.42	.9922	.0078	2.92	.9983	.0017	3.42	.9997	.0003	3.92	1.0000	.0000
2.43	.9925	.0075	2.93	.9983	.0017	3.43	.9997	.0003	3.93	1.0000	.0000
2.44	.9927	.0073	2.94	.9984	.0016	3.44	.9997	.0003	3.94	1.0000	.0000
2.45	.9929	.0071	2.95	.9984	.0016	3.45	.9997	.0003	3.95	1.0000	.0000
2.46	.9931	.0069	2.96	.9985	.0015	3.46	.9997	.0003	3.96	1.0000	.0000
2.47	.9932	.0068	2.97	.9985	.0015	3.47	.9997	.0003	3.97	1.0000	.0000
2.48	.9934	.0066	2.98	.9986	.0014	3.48	.9997	.0003	3.98	1.0000	.0000
2.49	.9936	.0064	2.99	.9986	.0014	3.49	.9998	.0002	3.99	1.0000	.0000
2.50	.9938	.0062	3.00	.9987	.0013	3.50	.9998	.0002	4.00	1.0000	.0000

APPENDIX D

CHI-SQUARE DISTRIBUTION

PERCENTAGE POINTS

$$F(\chi^2) = \int_0^{\chi^2} \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n-2}{2}} e^{-\frac{x}{2}} dx$$

n \ F	.005	.010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

APPENDIX E

GAMMA FUNCTION

Values of $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx;\ (n+1) = n\Gamma(n)$

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
1.00	1.00000	1.25	.90640	1.50	.88623	1.75	.91906
1.01	.99433	1.26	.90440	1.51	.88659	1.76	.92137
1.02	.98884	1.27	.90250	1.52	.88704	1.77	.92376
1.03	.98355	1.28	.90072	1.53	.88757	1.78	.92623
1.04	.97844	1.29	.89904	1.54	.88818	1.79	.92877
1.05	.97350	1.30	.89747	1.55	.88887	1.80	.93138
1.06	.96874	1.31	.89600	1.56	.88964	1.81	.93408
1.07	.96415	1.32	.89464	1.57	.89049	1.82	.93685
1.08	.95973	1.33	.89338	1.58	.89142	1.83	.93969
1.09	.95546	1.34	.89222	1.59	.89243	1.84	.94261
1.10	.95135	1.35	.89115	1.60	.89352	1.85	.94561
1.11	.94739	1.36	.89018	1.61	.89468	1.86	.94869
1.12	.94359	1.37	.88931	1.62	.89592	1.87	.95184
1.13	.93993	1.38	.88854	1.63	.89724	1.88	.95507
1.14	.93642	1.39	.88785	1.64	.89864	1.89	.95838
1.15	.93304	1.40	.88726	1.65	.90012	1.90	.96177
1.16	.92980	1.41	.88676	1.66	.90167	1.91	.96523
1.17	.92670	1.42	.88636	1.67	.90330	1.92	.96878
1.18	.92373	1.43	.88604	1.68	.90500	1.93	.97240
1.10	.92088	1.44	.88580	1.69	.90678	1.94	.97610
1.20	.91817	1.45	.88565	1.70	.90864	1.95	.97988
1.21	.91558	1.46	.88560	1.71	.91057	1.96	.98374
1.22	.91311	1.47	.88563	1.72	.91258	1.97	.98768
1.23	.91075	1.48	.88575	1.73	.91466	1.98	.99171
1.24	.90852	1.49	.88595	1.74	.91683	1.99	.99581
						2.00	1.00000

* For large positive values of x, $\Gamma(x)$ approximates the asymptotic series $x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left[1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right]$.

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